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A Mathematical Analysis of the Well-Posedness, Stability and Convergence of a Galerkin Reduced Order Model (ROM) for Coupled Fluid/Structure Interaction Problems

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Abstract

This document presents a mathematical analysis of the well-posedness, stability and convergence of a Galerkin/POD[1] Reduced Order Model (ROM) for coupled fluid/structure interaction problems. These results are an extension of the author's work during July - August 2007 [17], as well as the conference paper [4]. During the months of June - August 2008, the author:

- Derived sufficient conditions for stability and well-posedness of the solid wall boundary condition for the fluid ROM.
- Proved stability and well-posedness of the new acoustically-reflecting boundary condition on the solid wall.
- Expressed the acoustically-reflecting boundary condition in terms of the ROM coefficients and basis functions for the purpose of numerical implementation.
- Exhibited a penalty-like formulation that is equivalent to the usual weak implementation of the acoustically-reflecting boundary conditions and studied its stability.
- Showed stability of the coupled fluid/structure system under the new solid wall condition assuming a perturbed fluid pressure loading on the structure equations.
- Derived error estimates for the computed ROM solution relative to the CFD and the exact analytical solutions.
- Began to extend the said analysis to the more complicated situation of non-uniform base flow.

These derivations and proofs are presented in detail herein, to be ultimately condensed into a journal article.

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[1] Proper Orthogonal Decomposition; see Section 6.1.1 and Chapter 3 of [16].

1 Introduction

This document attempts to provide a rigorous analysis of the stability of a Reduced Order Model (ROM) of compressible fluid flow over a flat plate. It is an extension of the earlier works [2], [3], [4] and [17]. We focus on the stability and well-posedness of the solid-wall (plate) boundary condition, which has been changed from the no-penetration boundary condition formulated in [17] to the new, acoustically-reflecting boundary condition. The change in boundary condition was necessary due to practical difficulties with the former condition discovered upon numerical implementation of the ROM[2]. We are interested in not only the stability of the fluid equations, but also in the stability of the coupled fluid/solid system that arises when the new boundary condition on the plate is applied. This study of stability and well-posedness leads naturally to an analysis of the ROM's convergence.

For a thorough discussion of the problem formulation, the reader is referred to [2], [3], [4] and [17]. To keep this document self-contained, we briefly go over the notation and the key equations. Some of the more detailed results from [2], [3], [4] and [17] that are used or referenced herein can be found in the Appendix.

Let q denote the vector of fluid variables[3], split into a base state (denoted by $\bar{q}$) and a perturbation (denoted by q') component:

$$q = \bar{q} + q' = \begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \\ \bar{\zeta} \\ \bar{p} \end{pmatrix} + \begin{pmatrix} u' \\ v' \\ w' \\ \zeta' \\ p' \end{pmatrix} \tag{1}$$

Here, u, v and w are the three fluid velocity components, $\zeta = 1/\rho$ is the specific volume (where ρ is the density of the fluid), and p is the fluid pressure. The fluid variables are governed by the Euler equations, linearized about the base state $\bar{q}$, on an open bounded domain $\Omega \subset \mathbb{R}^3$. Partitioning the boundary of Ω into two boundaries, the far-field boundary ($\partial\Omega_F$) and the solid wall boundary ($\partial\Omega_P$),

$$\partial\Omega = \partial\Omega_F \cup \partial\Omega_P, \qquad \partial\Omega_F \cap \partial\Omega_P = \emptyset \tag{2}$$

the initial boundary value problem (IBVP) of interest is of the form[4]

$$\begin{array}{lll} \frac{\partial q'}{\partial t} + A_i \frac{\partial q'}{\partial x_i} + Cq' = 0, & \mathbf{x} \in \Omega, i = 1,2,3, & 0 < t < T \\ Pq' = h, & \mathbf{x} \in \partial\Omega_P, & 0 < t < T \\ Rq' = g, & \mathbf{x} \in \partial\Omega_F, & 0 < t < T \\ q'(\mathbf{x}, 0) = f(\mathbf{x}), & \mathbf{x} \in \Omega & \end{array} \tag{3}$$

where P and h specify the solid wall boundary conditions, R and g specify the far-field boundary conditions and $f : \Omega \to \mathbb{R}^5$ is a given function. The operators $\{A_i : i = 1,2,3\}$ and C are the following 5×5 matrices, derived in [12] and [19]:

$$A_1 = \begin{pmatrix} \bar{u} & 0 & 0 & 0 & \bar{\zeta} \\ 0 & \bar{u} & 0 & 0 & 0 \\ 0 & 0 & \bar{u} & 0 & 0 \\ -\bar{\zeta} & 0 & 0 & \bar{u} & 0 \\ \gamma\bar{p} & 0 & 0 & 0 & \bar{u} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \bar{v} & 0 & 0 & 0 & 0 \\ 0 & \bar{v} & 0 & 0 & \bar{\zeta} \\ 0 & 0 & \bar{v} & 0 & 0 \\ 0 & -\bar{\zeta} & 0 & \bar{v} & 0 \\ 0 & \gamma\bar{p} & 0 & 0 & \bar{v} \end{pmatrix}, \quad A_3 = \begin{pmatrix} \bar{w} & 0 & 0 & 0 & 0 \\ 0 & \bar{w} & 0 & 0 & 0 \\ 0 & 0 & \bar{w} & 0 & \bar{\zeta} \\ 0 & 0 & -\bar{\zeta} & \bar{w} & 0 \\ 0 & 0 & \gamma\bar{p} & 0 & \bar{w} \end{pmatrix} \tag{4}$$

$$C = \begin{pmatrix} \frac{\partial\bar{u}}{\partial x} & \frac{\partial\bar{u}}{\partial y} & \frac{\partial\bar{u}}{\partial z} & \frac{\partial\bar{p}}{\partial x} & 0 \\ \frac{\partial\bar{v}}{\partial x} & \frac{\partial\bar{v}}{\partial y} & \frac{\partial\bar{v}}{\partial z} & \frac{\partial\bar{p}}{\partial y} & 0 \\ \frac{\partial\bar{w}}{\partial x} & \frac{\partial\bar{w}}{\partial y} & \frac{\partial\bar{w}}{\partial z} & \frac{\partial\bar{p}}{\partial z} & 0 \\ \frac{\partial\bar{\zeta}}{\partial x} & \frac{\partial\bar{\zeta}}{\partial y} & \frac{\partial\bar{\zeta}}{\partial z} & -\left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z}\right) & 0 \\ \frac{\partial\bar{p}}{\partial x} & \frac{\partial\bar{p}}{\partial y} & \frac{\partial\bar{p}}{\partial z} & 0 & \gamma\left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z}\right) \end{pmatrix} \tag{5}$$

[2] See Section 2.

[3] This vector was denoted by U in [17].

[4] In (3) and from this point forward, we use the so-called Einstein summation convention: when an index appears twice in a single term, it implies we are summing over all possible values of that index $\left(\text{e.g., } A_i \frac{\partial q'}{\partial x_i} \equiv \sum_{i=1}^{3} A_i \frac{\partial q'}{\partial x_i}\right)$.

Remark 1: For the sake of brevity and to focus on the solid wall boundary condition, in many derivations and problem statements of the form (3), we intentionally omit the far-field boundary condition[5] on $\partial\Omega_F$. The results presented herein assume a stable and well-posed boundary condition has been imposed on $\partial\Omega_F$, so that stability and well-posedness of the IBVP (3) rests on the solid wall boundary condition. We refer the reader to [17] for a detailed discussion of the far-field conditions.

We will denote the coordinate vector interchangeably as $\mathbf{x}^T \equiv \begin{pmatrix} x & y & z \end{pmatrix}$ or $\mathbf{x}^T \equiv \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$. The orthonormal[6] vector basis for the fluid ROM is denoted by $\{\phi_k(\mathbf{x}) \in \mathbb{R}^5 : k = 1,\ldots,M\}$, so that the fluid variables expanded in this basis are

$$q' = \sum_{k=1}^{M} a_k(t)\phi_k(\mathbf{x}) \tag{6}$$

Here, the $a_k(t)$ are the fluid ROM coefficients to be solved for in the reduced order model. We denote the i^{th} component of $\phi_k(\mathbf{x})$ where $i \in \{1,2,3,4,5\}$ by $\phi_k^i(\mathbf{x})$.

In the implementation, Ω is taken to be a cube with sides of finite length $L > 0$: $\Omega = \Omega_x \times \Omega_y \times \Omega_z = (0,L) \times (0,L) \times (0,L)$. The flat plate over which the fluid flows is in the $z = 0$ plane, meaning

$$\partial\Omega_P = \Omega_x \times \Omega_y \times \{z = 0\} \tag{7}$$

and the outward unit normal to $\partial\Omega_P$ is $\mathbf{n}^T = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$. In the remainder of this document, we will use the terms "plate boundary" and "solid wall boundary" interchangeably. Note, however, that, unless stated otherwise, the stability and well-posedness results presented in this work hold for *any* open, bounded $\Omega \subset \mathbb{R}^3$ and *any* boundary $\partial\Omega_P$ of Ω.

We *do* make several assumptions on which the theoretical results discussed herein rest. From this point forward (unless indicated otherwise), assume that:

$$\begin{array}{ll} 1. & \text{The base flow satisfies a no-penetration boundary condition: } \bar{u}_n \equiv \bar{u}n_1 + \bar{v}n_2 + \bar{w}n_3 \equiv 0 \text{ on } \partial\Omega_P \\ 2. & \text{The base flow is uniform: } \nabla\bar{q} \equiv 0 \end{array} \tag{8}$$

In 1., $\mathbf{n}^T = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}$ is the outward unit normal to $\partial\Omega_P$. A direct consequence of 2. is that

$$\nabla\bar{q} \equiv 0 \quad \Rightarrow \quad \begin{cases} \frac{\partial A_i}{\partial x_j} \equiv 0 \text{ for } i,j \in \{1,2,3\} \\ C \equiv 0 \end{cases} \tag{9}$$

where A_i and C are the operators defined in (4) and (5) respectively.

We now turn our attention to the structure side, namely the plate equations. Rather than keeping the plate stationary, we will allow it to deform slightly. Assume that the deformations are restricted to the direction normal to the plate, leading to a non-zero displacement only in the z-direction. We will denote this z-displacement by $\eta = \eta(x,y,t)$. The displacement η is governed by the von Karman equation:

$$\begin{array}{lll} \rho_s h \frac{\partial^2\eta}{\partial t^2} + D_{bend}(\nabla^4\eta) = g, & (x,y) \in \Omega_x \times \Omega_y, & 0 < t < T \\ \eta(x,y,t) = 0, & (x,y) \in (\partial\Omega_x \times \Omega_y) \cup (\Omega_x \times \partial\Omega_y), & 0 < t < T \\ \frac{\partial^2\eta}{\partial x^2}(x,y,t) = 0, & (x,y) \in (\partial\Omega_x \times \Omega_y) \cup (\Omega_x \times \partial\Omega_y), & 0 < t < T \end{array} \tag{10}$$

In (10), h is the thickness of the plate, ρ_s is the density of the plate and $D_{bend} = \frac{Eh^3}{12(1-\nu^2)}$ is the bending stiffness[7]. The function g is the unsteady fluid pressure loading, so that

$$g(x,y,t) = -p'(x,y,0,t) \tag{11}$$

[5] Recall that the far-field boundary condition specified on $\partial\Omega_F$ is the no-reflecting condition, formulated in Section 2.3 of [17].

[6] Orthonormal in the so-called (H,Ω)-norm, defined in Section 2.1.

[7] Here, E is Young's modulus and ν is Poisson's ratio; see Section 3.1 of [17].

(11) couples the structure equations (10) with the fluid equations (3). In the structure ROM, the z-displacement is expanded in a scalar orthonormal[8] ROM basis $\{\xi_k(x,y) : k = 1,\ldots,P\}$ as follows:

$$\eta = \sum_{k=1}^{P} b_k(t)\xi_k(x,y) \tag{12}$$

The solid wall boundary conditions on the fluid variables will further couple the fluid and structure equations.

Having given a brief overview of the equations and problem statement, we are ready to proceed to the analysis. The remainder of this paper is organized as follows. In Section 2, we derive sufficient conditions for a set of boundary conditions on $\partial\Omega$ to be well-posed and stable for the fluid system (3). We then formulate the new acoustically-reflecting boundary condition on the plate and show its well-posedness and stability. In Section 3, a penalty-like formulation of the acoustically-reflecting boundary condition is uncovered and analyzed. Section 4 deals with the implementation of the new boundary condition: the condition is expressed in terms of the ROM coefficients and basis functions, which gives rise to yet another penalty-like expression. The resulting coupled fluid/structure system for the ROM coefficients is examined in Section 5, and its stability assuming a perturbed fluid pressure loading on the plate is shown. Error bounds for the computed ROM solution relative to the CFD solution and (via the triangle inequality) the exact analytical solution are derived in Section 6. Section 7 contains a brief discussion of how to extend the well-posedness and stability results presented herein to the case of non-uniform base flow. Conclusions are offered in Section 8. Section 9 is the Appendix, which summarizes many of the mathematical tools used in the analysis.

2 Stability and Well-Posedness of Plate Boundary Condition for the Fluid Equations

2.1 Well-Posedness: a General Analysis for the Linearized Euler Equations

Let us denote

$$u_n \equiv \mathbf{u}\cdot\mathbf{n} = un_1 + vn_2 + wn_3 \tag{13}$$

and

$$A_n \equiv \mathbf{A}\cdot\mathbf{n} \equiv A_1n_1 + A_2n_2 + A_3n_3 \tag{14}$$

As stated in the Introduction, we will assume $\bar{u}_n = 0$ and uniform base flow $\left(\frac{\partial A_n}{\partial x_i} \equiv 0 \text{ for } i = 1,2,3 \text{ and } C \equiv 0\right)$. Recall the matrix H[9], the symmetrizer of the linearized Euler equations (3):

$$H = \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & \bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho} & 0 & 0 \\ 0 & 0 & 0 & \alpha^2\gamma\bar{\rho}^2\bar{p} & \bar{\rho}\alpha^2 \\ 0 & 0 & 0 & \bar{\rho}\alpha^2 & \frac{(1+\alpha^2)}{\gamma\bar{p}} \end{pmatrix} \tag{15}$$

H is symmetric positive definite and has the property that the matrices $\{HA_i : i = 1,2,3\}$ are also symmetric. Since H is symmetric positive definite, one can define with respect to it an inner product and a norm[10]. For any symmetric positive definite matrix M, we will denote the (M,Ω)-inner product and (M,Ω)-norm by:

$$(u,v)_{(M,\Omega)} \equiv \int_\Omega u^T M v d\Omega, \qquad ||u||^2_{(M,\Omega)} \equiv (u,u)_{(M,\Omega)} \tag{16}$$

[8]Orthonormal in the $L^2(\partial\Omega_P)$ norm.
[9]See [2] or Section 9.7 of the Appendix.
[10]See [2] and [17].

Similarly, define the $\langle M, \partial\Omega \rangle$-inner product and $\langle M, \partial\Omega \rangle$-norm by

$$\langle u, v \rangle_{(M,\partial\Omega)} \equiv \int_{\partial\Omega} u^T M v dS, \qquad ||u||^2_{(M,\partial\Omega)} \equiv \langle u, u \rangle_{(M,\partial\Omega)} \tag{17}$$

Note that

$$(u, v)_{(M,\Omega)} = (M^{1/2} u, M^{1/2} v)_{L^2(\Omega)} \tag{18}$$

where

$$(u, v)_{L^2(\Omega)} \equiv \int_\Omega u^T v d\Omega \tag{19}$$

is the usual L^2 inner product on Ω (and similarly for $\partial\Omega_P$) and $M^{1/2}$ is the "square root" factor of M, which exists since M is assumed to be positive definite.

Remark 2: Recall that the fluid ROM basis functions (modes) $\{\phi_i : i = 1, \ldots, M\}$ are orthonormal with respect to the (H, Ω)-norm, and that the equations (3) are projected onto these modes using the (H, Ω)-inner product. This inner product is selected over the usual L^2 inner product to ensure a stable Galerkin approximation; see Section 3 of [4].

The following theorem gives sufficient conditions on the boundary conditions for the IBVP (3) to be well-posed[11]. We use the energy approach to study well-posedness: an IBVP is well-posed if the energy associated with the analogous homogeneous IBVP (that is, the original IBVP but with homogeneous Dirichlet boundary conditions and no source term) is non-increasing. The notation from this point onward (in the context of the IBVP (3)) is as follows:

$$\begin{aligned} q'_b &\equiv \{q \in \mathbb{R}^5 : q' - q = 0 \text{ and } Pq' - h = 0 \text{ on } \partial\Omega_P\} \\ q'_f &\equiv \{q \in \mathbb{R}^5 : q' - q = 0 \text{ and } Rq' - g = 0 \text{ on } \partial\Omega_F\} \end{aligned} \tag{20}$$

$$\begin{aligned} q'_{b0} &\equiv \{q \in \mathbb{R}^5 : q' - q = 0 \text{ and } Pq' = 0 \text{ on } \partial\Omega_P\} \\ q'_{f0} &\equiv \{q \in \mathbb{R}^5 : q' - q = 0 \text{ and } Rq' = 0 \text{ on } \partial\Omega_F\} \end{aligned} \tag{21}$$

In other words, q'_b is the vector of boundary conditions on $\partial\Omega_P$ as specified in (3) and q'_{b0} is the vector of boundary conditions on $\partial\Omega_P$ as specified in (3) but with $h = 0$ (and similarly for q'_f and q'_{f0} on Ω_F).

Theorem 2.1.1. *Assuming a uniform base flow ($\bar{u}_n = 0, \nabla\bar{q} = 0$), the IBVP* (3) *for the linearized Euler equations is well-posed if*

$$\frac{1}{2}\int_{\partial\Omega_P} q'^T_{b0} H A_n q'_{b0} dS \geq 0 \tag{22}$$

and

$$\frac{1}{2}\int_{\partial\Omega_F} q'^T_{f0} H A_n q'_{f0} dS \geq 0 \tag{23}$$

Here $\partial\Omega \equiv \partial\Omega_P \cup \partial\Omega_F$, with $\partial\Omega_P \cap \partial\Omega_F = \emptyset$; in context, $\partial\Omega_P$ is the solid-wall (or plate) boundary and $\partial\Omega_F$ is the far-field boundary of the open bounded domain $\Omega \subset \mathbb{R}^3$.

Proof. By Definition 2.8 in [14] (repeated in Section 9.9 of the Appendix for easy reference), to show well-posedness of (3), it is sufficient to show well-posedness of this IBVP with $h = g = 0$. We prove well-posedness by demonstrating that the energy in the (H, Ω)-norm is non-increasing. To go from the third to the fourth line of (24), we apply the integration by parts "trick", found in the Section 9.1 of the Appendix. This is possible because $\{HA_i : i = 1, 2, 3\}$ are

[11] Refer to Section 9.9 in the Appendix for formal definitions of well-posedness, quoted from [14].

all symmetric.

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||q'||^2_{(H,\Omega)} &= \frac{1}{2}\frac{d}{dt}\int_\Omega q'^T H q' d\Omega \\
&= \int_\Omega q'^T H \frac{\partial q'}{\partial t} d\Omega \\
&= -\int_\Omega q'^T H A_i \frac{\partial q'}{\partial x_i} d\Omega \\
&= -\frac{1}{2}\int_\Omega \left[\frac{\partial}{\partial x_i}(q'^T H A_i q') - q'^T \underbrace{\frac{\partial (HA_i)}{\partial x_i}}_{\equiv 0 \text{ (uniform base flow)}} q' \right] d\Omega \qquad (24)\\
&= -\frac{1}{2}\int_\Omega \frac{\partial}{\partial x_i}(q'^T H A_i q') d\Omega \\
&= -\frac{1}{2}\int_{\partial\Omega} q'^T H A_n q' dS \\
&= -\frac{1}{2}\int_{\partial\Omega_F} q'^T_{b0} H A_n q'_{b0} dS - \frac{1}{2}\int_{\partial\Omega_P} q'^T_{f0} H A_n q'_{f0} dS \\
&\leq 0
\end{aligned}$$

if conditions (22) and (23) hold. □

Theorem 2.1.1 enables one to determine if a set of boundary conditions on a boundary of Ω is well-posed for (3). Note, however, that it is a sufficient but *not* a necessary condition, meaning an IBVP can be well-posed even if (22) and (23) fail.

2.2 Special Case: Well-Posedness Analysis when $\Omega = (0,L_1)\times(0,L_2)\times(0,L_3)$

It turns out that there is a particularly easy way to check the conditions (22) and (23) when $\Omega \subset \mathbb{R}^3$ is a box[12]. In this specific case, it is convenient to do the well-posedness analysis in the characteristic variables $V' = S^{-1}q'$. Here, S is the matrix that diagonalizes A_n (14), so that

$$A_n = S\Lambda S^{-1} \qquad (25)$$

where $\Lambda = \text{diag}\{\bar{u}_n, \bar{u}_n, \bar{u}_n, \bar{u}_n + c, \bar{u}_n - c\}$. The matrices S and S^{-1} were derived in [17] and can be found in Section 9.7.2 of the Appendix. In the characteristic variables, the linearized Euler equations are

$$\frac{\partial V'}{\partial t} + S^{-1}A_i S \frac{\partial V'}{\partial x_i} = 0 \qquad (26)$$

When $\Omega = \Omega_{x_1}\times\Omega_{x_2}\times\Omega_{x_3}$ is a box, its boundaries are simply planes. The six faces (boundaries) of Ω along with their outward unit normals are listed in the following table:

Boundary	Notation	$\mathbf{n}^T$
$\Omega_{x_1}\times\Omega_{x_2}\times\{x_3=0\}$	$\partial\Omega_3^-$	$\begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$
$\Omega_{x_1}\times\Omega_{x_2}\times\{x_3=L_3\}$	$\partial\Omega_3^+$	$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$
$\Omega_{x_1}\times\Omega_{x_3}\times\{x_2=0\}$	$\partial\Omega_2^-$	$\begin{pmatrix} 0 & -1 & 0 \end{pmatrix}$
$\Omega_{x_1}\times\Omega_{x_3}\times\{x_2=L_2\}$	$\partial\Omega_2^+$	$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$
$\Omega_{x_2}\times\Omega_{x_3}\times\{x_1=0\}$	$\partial\Omega_1^-$	$\begin{pmatrix} -1 & 0 & 0 \end{pmatrix}$
$\Omega_{x_2}\times\Omega_{x_3}\times\{x_1=L_1\}$	$\partial\Omega_1^+$	$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

In general, we denote

$$\partial\Omega_i^\pm \equiv \{\text{side of } \Omega \text{ normal to the } i^{th} \text{ axis with an outward unit normal in the } \pm\, i^{th} \text{ direction} : i = 1,2,3\} \qquad (27)$$

Let S_i be the matrix that diagonalizes A_i for $i = 1,2,3$ (so that $S_i^{-1}A_iS_i = \Lambda_i$) and observe that

$$S_i^{-1}\mathbf{A}\cdot\mathbf{e}_i S_i = S^{-1}A_iS_i = \Lambda_i \qquad (28)$$

[12]Or, more generally, when the unit normals to the boundaries of Ω are not spatially varying; see Remark 4.

Here, $\mathbf{e}_i \in \mathbb{R}^3$ is the unit vector in the positive x_i-direction and $\mathbf{A} \equiv \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}$. That is, S_i diagonalizes A_n when $\mathbf{n} = \mathbf{e}_i$. Let Q be the symmetric, positive definite, diagonal matrix that simultaneously symmetrizes[13] $\{S^{-1}A_iS : i = 1,2,3\}$:

$$Q \equiv \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \tag{29}$$

and denote

$$A_i^S \equiv QS^{-1}A_iS \tag{30}$$

We will use the (Q,Ω)-norm to show well-posedness, again by showing that the energy, this time in the (Q,Ω)-norm, is non-increasing.

Theorem 2.2.1. *Consider the linearized Euler equations in the characteristic variables V' (26) on $\Omega = \Omega_x \times \Omega_y \times \Omega_z = (0,L_1) \times (0,L_2) \times (0,L_3)$ with $\bar{u}_n = 0$ and $\nabla \bar{q} = 0$. A boundary condition on $\partial\Omega_i^+$ is well-posed if*

$$\frac{1}{2}[V_{b0}'^T A_i^S V_{b0}']_{x_i=L_i} \geq 0 \tag{31}$$

for $i = 1,2,3$. A boundary condition on $\partial\Omega_i^-$ is well-posed if

$$\frac{1}{2}[V_{b0}'^T A_i^S V_{b0}']_{x_i=0} \leq 0 \tag{32}$$

for $i = 1,2,3$. Here, V_{b0}' is the vector of boundary conditions prescribed on $\partial\Omega_i^\pm$, but homogenized (so that, for example, if $PV' - h = 0$ is prescribed on $\partial\Omega_i^+$, then $V_{b0}' = \{V \in \mathbb{R}^5 : V' - V = 0 \text{ and } PV' = 0 \text{ on } \partial\Omega_i^+\}$).

Proof. We show well-posedness by showing that the energy of (26) in the (Q,Ω)-norm assuming homogeneous boundary conditions is non-increasing. Consider (26) on $\Omega = \Omega_x \times \Omega_y \times \Omega_z = (0,L_1) \times (0,L_2) \times (0,L_3)$ but with $h = g = 0$. Then

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}||V'||^2_{(Q,\Omega)} &= \tfrac{1}{2}\tfrac{d}{dt}\int_\Omega V'^T Q V' d\Omega \\
&= \int_\Omega V'^T Q \tfrac{\partial V'}{\partial t} d\Omega \\
&= -\int_\Omega V'^T A_i^S \tfrac{\partial V'}{\partial x_i} d\Omega \\
&= -\tfrac{1}{2}\int_\Omega \left[\tfrac{\partial}{\partial x_i}(V'^T A_i^S V') - V'^T \underbrace{\frac{\partial A_i^S}{\partial x_i}}_{\equiv 0 \text{ (uniform base flow)}} V' \right] d\Omega \\
&= -\tfrac{1}{2}\int_\Omega \tfrac{\partial}{\partial x_i}(V'^T A_i^S V') d\Omega \\
&= -\tfrac{1}{2}\int_{\Omega_{x_1}}\int_{\Omega_{x_2}} [V_{b0}'^T A_3^S V_{b0}']_{x_3=0}^{x_3=L_3} d\Omega_{x_2} d\Omega_{x_1} - \tfrac{1}{2}\int_{\Omega_{x_2}}\int_{\Omega_{x_3}} [V_{b0}'^T A_1^S V_{b0}']_{x_1=0}^{x_1=L_1} d\Omega_{x_3} d\Omega_{x_2} \\
&\quad -\tfrac{1}{2}\int_{\Omega_{x_1}}\int_{\Omega_{x_3}} [V_{b0}'^T A_2^S V_{b0}']_{x_2=0}^{x_2=L_2} d\Omega_{x_3} d\Omega_{x_1} \\
&= \int_{\Omega_{x_1}}\int_{\Omega_{x_2}} \left(-\tfrac{1}{2}[V_{b0}'^T A_3^S V_{b0}']_{x_3=L_3} + \tfrac{1}{2}[V_{b0}'^T A_3^S V_{b0}']_{x_3=0} \right) d\Omega_{x_2} d\Omega_{x_1} \\
&\quad + \int_{\Omega_{x_2}}\int_{\Omega_{x_3}} \left(-\tfrac{1}{2}[V_{b0}'^T A_1^S V_{b0}']_{x_1=L_1} + \tfrac{1}{2}[V_{b0}'^T A_1^S V_{b0}']_{x_1=0} \right) d\Omega_{x_3} d\Omega_{x_2} \\
&\quad + \int_{\Omega_{x_1}}\int_{\Omega_{x_3}} \left(-\tfrac{1}{2}[V_{b0}'^T A_2^S V_{b0}']_{x_2=L_2} + \tfrac{1}{2}[V_{b0}'^T A_2^S V_{b0}']_{x_2=0} \right) d\Omega_{x_3} d\Omega_{x_1}
\end{aligned} \tag{33}$$

If (31) holds on Ω_i^+ and (32) holds on Ω_i^- for all $i = 1,2,3$, then the last line in (33) is non-positive, which implies that the boundary conditions are well-posed. ☐

Remark 3: We emphasize that Theorem 2.2.1 is a sufficient but *not* a necessary condition for well-posedness. One could have, for instance, the expression (32) > 0 on some boundary $\partial\Omega_i^-$ but still have $\frac{1}{2}\frac{d}{dt}||V'||^2_{(Q,\Omega)} \leq 0$ for the net set

[13] This matrix is derived in Section 9.8 of the Appendix (229).

of boundary conditions as long as their energy contribution is sufficiently negative to balance out the positive energy contribution from the boundary condition on $\partial\Omega_i^+$.

As implied by Remark 3, Theorem 2.2.1 is useful in analyzing boundary conditions one boundary at a time. Suppose one of the boundaries is a solid wall boundary and the remaining five boundaries are far-field boundaries on which one knows that a set of well-posed conditions are being imposed. Then one may check the well-posedness of the IBVP by using the theorem to check the well-posedness of the wall condition independently of the far-field conditions.

Theorem 2.2.1 gives rise to the following corollary that further facilitates the task of checking well-posedness in the case of a domain that is simply a box.

Corollary 2.2.2. *Consider again the linearized Euler equations in the characteristic variables* V' (26) *on* $\Omega = \Omega_x \times \Omega_y \times \Omega_z = (0,L_1)\times(0,L_2)\times(0,L_3)$ *with* $\bar{u}_n = 0$ *and* $\nabla\bar{q} = 0$*. Let* λ_j^i *be the* j^{th} *eigenvalue of* A_i*. A boundary condition on* $\partial\Omega_i^+$ *is well-posed if*

$$\frac{1}{2}\left[\sum_{j=1}^{5}\lambda_j^i[(V'_{b0})_j]^2\right]_{x_i=L_i} \geq 0 \tag{34}$$

for $i = 1,2,3$*. A boundary condition on* $\partial\Omega_i^-$ *is well-posed if*

$$\frac{1}{2}\left[\sum_{j=1}^{5}\lambda_j^i[(V'_{b0})_j]^2\right]_{x_i=0} \leq 0 \tag{35}$$

for $i = 1,2,3$*. Here,* V'_{b0} *is the homogeneous variant of the boundary condition on* $\partial\Omega_i^\pm$ *(see* (27)*).*

Proof. Without loss of generality, we will show that (31) is equivalent to (34); from this it will be clear that (32) is equivalent to (35). The left-hand-side of (31) can be rewritten as

$$\frac{1}{2}\left[V'^T_{b0}A_i^S V'_{b0}\right]_{x_i=L_i} = \frac{1}{2}\left[V'^T_{b0}QS_i^{-1}S_i\Lambda_i S_i^{-1}S_i V'_{b0}\right]_{x_i=L_i} = \frac{1}{2}\left[V'^T_{b0}Q\Lambda_i V'_{b0}\right]_{x_i=L_i} \tag{36}$$

Because Q is positive definite, it has a "square root" factor: $Q = Q^{1/2}Q^{1/2}$. Using this factorization, an equivalent way of writing condition (36) is to require the boundary conditions on $\partial\Omega_i^+$ to satisfy

$$\frac{1}{2}(Q^{1/2})^T\left[V'^T_{b0}\Lambda_i V'_{b0}\right]_{x_i=L_i} Q^{1/2} \geq 0 \tag{37}$$

or

$$\frac{1}{2}\left[\sum_{j=1}^{5}\lambda_j^i[(V'_{b0})_j]^2\right]_{x_i=L_i} \geq 0 \tag{38}$$

which is precisely (34). The proof for $\partial\Omega_i^-$ is essentially identical, so we do not repeat it here. □

Remark 4: One may ask why Theorem 2.2.1 and Corollary 2.2.2 require that $\Omega \subset \mathbb{R}^3$ be a box. Actually, the only property Ω needs to satisfy for these results to hold is it must have a constant (that is, not spatially-varying) normal $\mathbf{n}$ to all its boundaries $\partial\Omega$; so, for instance, the results would hold if Ω were a rotated box.

As an example, and for later reference, consider the model problem mentioned in the Introduction and in [17] in which the non-free surface is a flat plate in the $z = 0$ plane. This boundary corresponds to $\partial\Omega_3^-$, meaning a sufficient condition for well-posed plate boundary conditions on this side is (35) with $i = 3$:

$$\frac{1}{2}\left[\sum_{j=1}^{5}\lambda_j^3[(V'_{b0})_j]^2\right]_{x_3=0} \leq 0 \tag{39}$$

Here, the λ_j^3 are the eigenvalues of $\Lambda_3 = S_3 A_3^S S_3^{-1}$, with

$$S_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{\bar{\zeta}}{2c} & -\frac{\bar{\zeta}}{2c} \\ 0 & 0 & 0 & \frac{\gamma p}{2c} & \frac{\gamma p}{2c} \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & c & \\ & & & & -c \end{pmatrix}, \quad S_3^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\bar{\zeta}}{\gamma p} \\ 0 & 0 & 1 & 0 & \frac{c}{\gamma p} \\ 0 & 0 & -1 & 0 & \frac{c}{\gamma p} \end{pmatrix} \tag{40}$$

2.3 Acoustically-Reflecting Plate Boundary Condition

In [17], the boundary condition enforced on the fluid variables at the plate boundary was the linearized version of the no-penetration boundary condition (BC), $\mathbf{u} \cdot \mathbf{n} = -\dot{\eta}$:

$$u_n' + \bar{\mathbf{u}} \cdot \nabla \eta = -\dot{\eta} \quad \text{on} \quad \partial\Omega_P \tag{41}$$

Here, the "$\cdot$" operator represents a time derivative, i.e., $\dot{\eta} \equiv \frac{\partial \eta}{\partial t}$ and

$$\mathbf{u}^T \equiv \begin{pmatrix} u & v & w \end{pmatrix} \tag{42}$$

The no-penetration condition (41) was implemented weakly according to Algorithm 1 below[14].

One can show that the linearized no-penetration condition (41) is well-posed and stable for the fluid ROM (Theorem 2.5.1). Unfortunately, numerical experiments suggest that enforcing the condition by including (46) in the j^{th} ROM equation is too weak. On a simple benchmark problem with a stationary plate, it was found that the implementation described in Algorithm 1 does not effectively enforce $u_n' = 0$ at the plate boundary, as it should.

A condition that turns out to be mathematically equivalent[15] to (41) but that does not suffer from this problem is the so-called acoustically-reflecting boundary condition. This new condition is posed using the characteristic decomposition. Since we are assuming that $\bar{u}_n \equiv 0$, it follows that the characteristic speeds are $\{0,0,0,c,-c\}$. In particular, the fourth characteristic is outgoing and the fifth characteristic is incoming. For a stationary wall, the so-called perfectly-reflecting boundary condition says to set the incoming characteristic, V_5', equal to the outgoing characteristic, V_4'. When the wall velocity is $u_b' \equiv u_b'(x,y,t)$, the condition amounts to setting

$$V_5' = V_4' - 2u_b' \quad \text{on} \quad \partial\Omega_P \tag{47}$$

Since the characteristics with wave speed $\bar{u}_n \pm c$ are acoustic waves (whereas the others are entropic/vortical), we will call the "perfectly-reflecting condition" an "acoustically-reflecting" boundary treatment. Condition (47) can be written in matrix form as

$$P^S V' = h^S \quad \text{on} \quad \partial\Omega_P \tag{48}$$

with

$$P^S = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & -1 & 1 \end{pmatrix}, \quad h^S = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2u_b' \end{pmatrix} \tag{49}$$

[14] See Section 4 for more on weak implementations of solid wall boundary conditions for the fluid ROM; refer also to [17].

[15] Using the fact that $V_4' = -u_n' + \frac{c}{\gamma p} p'$ and $V_5' = u_n' + \frac{c}{\gamma p} p'$, (47) translates to $u_n' = u_b'$ in the original variables, which is precisely the no-penetration condition (41).

Algorithm 1 Weak Implementation of the No-Penetration Boundary Condition (41) using Integration by Parts (IBP)

1. Project the first line of of (3) onto the j^{th} POD mode using the (H,Ω)-inner product:

$$\left(\phi_j, \frac{\partial q'}{\partial t}\right)_{(H,\Omega)} = -\left(\phi_j, A_i \frac{\partial q'}{\partial x_i}\right)_{(H,\Omega)} \tag{43}$$

2. Integrate the second term in (43) by parts and substitute $q' \leftarrow q'_b$ into the boundary integral over $\partial\Omega_P$:

$$\left(\phi_j, \frac{\partial q'}{\partial t}\right)_{(H,\Omega)} = -\underbrace{\int_{\partial\Omega_P} \phi_j^T H A_n q'_b dS}_{\equiv I_{P_j}} - \int_{\partial\Omega_F} \phi_j^T H A_n q' dS + \int_{\Omega} \frac{\partial}{\partial x_i}[\phi_j^T H A_i] q' d\Omega \tag{44}$$

Here, q'_b is the vector specifying the condition (41) on $\partial\Omega_P$, so that

$$HA_n q'_b = \begin{pmatrix} n_1 p' \\ n_2 p' \\ n_3 p' \\ 0 \\ -\bar{\mathbf{u}} \cdot \nabla\eta - \dot{\eta} \end{pmatrix} \tag{45}$$

3. Compute the boundary integral term appearing in the j^{th} ROM equation (I_{P_j}): substitute $-\bar{\mathbf{u}} \cdot \nabla\eta - \dot{\eta} \leftarrow u'_b$ and the expansion $p' \leftarrow \sum_{k=1}^{M} a_k \phi_k^5$ into the last and first three components of (45) respectively; then substitute (45) into the $\partial\Omega_P$ contribution of (44), to get

$$I_{P_j} = \sum_{k=1}^{M} a_k(t) \left[\int_{\partial\Omega_P} (\phi_j^n \phi_k^5 + \phi_j^5 u'_b) dS\right] \tag{46}$$

2.4 Well-Posedness of Acoustically-Reflecting Boundary Condition

We will use the analysis in Sections 2.1 and 2.2 to show that the acoustically-reflecting boundary condition is well-posed for the IBVP (3). The following lemma for $\Omega = (0,L_1)\times(0,L_2)\times(0,L_3)$ demonstrates how simple it is to check well-posedness for this special case using Corollary 2.2.2.

Lemma 2.4.1. *Assume $\bar{u_n} = 0$, $\nabla\bar{q} = 0$, and $\Omega = (0,L_1)\times(0,L_2)\times(0,L_3)$. Then the acoustically-reflecting boundary condition* (47) *for* (3) *on any boundary $\{\partial\Omega_i^{\pm} : i = 1,2,3\}$ is well-posed.*

Proof. To show well-posedness, we need only consider the homogeneous problem. Corollary 2.2.2 requires that

$$\begin{aligned} 0 &\geq \tfrac{1}{2}\textstyle\sum_{i=1}^{5}\lambda_i[(V'_{b0})_i]^2 \\ &= c(V'_4)^2 - c(V'_4 - 2u'_b)^2 \\ &= 4cu'_b\left[V'_4 - u'_b\right] \end{aligned} \tag{50}$$

The right-hand-side of (50) is identically 0 if $u'_b = 0$ (since $\lambda_4 = c$ and $\lambda_5 = -c$ when $\bar{u_n} = 0$). (50) implies that $||V'(\cdot,T)||^2_{(Q,\Omega)} \leq K = const$. By Definition 2.8 in [14] (see Section 9.9 in the Appendix), the acoustically-reflecting boundary condition on $\partial\Omega_i^{\pm}$ is well-posed. □

Remark 5: Note that Lemma 2.4.1 shows that the acoustically-reflecting boundary condition (47) is well-posed *even for $u'_b \neq 0$*. One can set $u'_b = 0$ when computing the energy estimate because it is sufficient to consider the homogeneous case according to the formal definition of stability in [14].

It turns out that one can show a stronger result regarding the well-posedness of (47), namely that it is *strongly* well-posed[16] on *any* open bounded $\Omega \subset \mathbb{R}^3$. This result is proven in the following theorem. Since strong well-posedness implies regular well-posedness, Theorem 2.4.2 shows that (47) is well-posed *and* strongly well-posed for *any* shape $\partial\Omega_P$.

Theorem 2.4.2. *Assume $\bar{u_n} = 0$, $\nabla\bar{q} = 0$ and let $\Omega \subset \mathbb{R}^3$ be an open bounded domain. Then the acoustically-reflecting boundary condition* (47) *for* (3) *on any boundary $\partial\Omega_P$ is strongly well-posed.*

Proof. Consider the (H^2,Ω)-norm, that is $||\cdot||_{(H^2,\Omega)}$. It is easy to see that this defines a valid norm since H^2 is positive definite and symmetric, a consequence of the symmetry and positive-definiteness of H. For the upcoming analysis, we return to the original variables q'. Then, writing $q'_b = SV'_b$ where V'_b is the vector specifying the solid wall boundary condition in the characteristic variables,

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}||q'||^2_{(H^2,\Omega)} &= \tfrac{1}{2}\tfrac{d}{dt}\int_\Omega q'^T H^2 q' d\Omega \\ &= \int_\Omega q'^T H^2 \tfrac{\partial q'}{\partial t} d\Omega \\ &= -\int_\Omega q'^T H^2 A_i \tfrac{\partial q'}{\partial x_i} d\Omega \\ &= -\tfrac{1}{2}\int_\Omega \left[\tfrac{\partial}{\partial x_i}(q'^T H^2 A_i q') - q'^T \underbrace{\frac{\partial (H^2 A_i)}{\partial x_i}}_{\equiv 0 \text{ (uniform mean flow)}} q'\right] d\Omega \\ &= -\tfrac{1}{2}\int_\Omega \tfrac{\partial}{\partial x_i}(q'^T H^2 A_i q') d\Omega \\ &= -\tfrac{1}{2}\int_{\partial\Omega_P} q_b'^T H^2 A_n q'_b dS \end{aligned} \tag{51}$$

Integrating (51) from 0 to T gives

$$||q'(\cdot,T)||^2_{(H^2,\Omega)} \leq ||q'(\cdot,0)||^2_{(H^2,\Omega)} - \int_0^T \left(\int_{\partial\Omega_P} q_b'^T H^2 A_n q'_b dS\right) dt \tag{52}$$

[16] See Definition 2.9 in [14], repeated in Section 9.9 of the Appendix.

For the acoustically-reflecting boundary condition,

$$V_b' = \begin{pmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \\ V_4' - 2u_b' \end{pmatrix} \tag{53}$$

Using (53), one can easily compute that

$$H^2 A_n q_b' = H^2 S \Lambda V_b' = \begin{pmatrix} \bar{\rho}^2 c n_1 (V_4' - u_b') \\ \bar{\rho}^2 c n_2 (V_4' - u_b') \\ \bar{\rho}^2 c n_3 (V_4' - u_b') \\ \alpha^2 \bar{\rho} u_b' \\ \frac{(1+\alpha^2)}{\gamma \bar{p}} u_b' \end{pmatrix} \tag{54}$$

Also,

$$q_b' = S \Lambda V_b' = \begin{pmatrix} c n_1 (V_4' - u_b') \\ c n_2 (V_4' - u_b') \\ c n_3 (V_4' - u_b') \\ -\bar{\zeta} u_b' \\ \gamma \bar{p} u_b' \end{pmatrix} \tag{55}$$

so that

$$q_b'^T H^2 A_n q_b' = \bar{\rho}^2 c^2 (V_4' - u_b')^2 + u_b'^2 \tag{56}$$

Now, from (55),

$$|q_b'|^2 = q_b'^T q_b' = c^2 (V_4' - u_b')^2 + (\bar{\zeta}^2 + \gamma^2 \bar{p}^2) u_b'^2 \tag{57}$$

so that, using the relation that $c^2 = \frac{\gamma \bar{p}}{\bar{\rho}}$,

$$\bar{\rho}^2 |q_b'|^2 = \bar{\rho}^2 c^2 (V_4' - u_b')^2 + (1 + (\bar{\rho} c)^4) u_b'^2 \tag{58}$$

meaning

$$q_b'^T H^2 A_n q_b' = \bar{\rho}^2 |q_b'|^2 - (\bar{\rho} c)^4 u_b'^2 \tag{59}$$

Substituting (59) into (52), one obtains the following estimate:

$$\begin{aligned} ||q'(\cdot,T)||^2_{(H^2,\Omega)} &\leq ||f(\cdot)||^2_{(H^2,\Omega)} + \int_0^T \left\{ \int_{\partial\Omega_P} \left(-\bar{\rho}^2 |q_b'|^2 + (\bar{\rho} c)^4 u_b'^2 \right) \right\} dt \\ &= ||f(\cdot)||^2_{(H^2,\Omega)} + \int_0^T \left(-||\bar{\rho} q_b'||^2_{L^2(\partial\Omega_P)} + ||(\bar{\rho} c)^2 u_b'||^2_{L^2(\partial\Omega_P)} \right) dt \\ &\leq ||f(\cdot)||^2_{(H^2,\Omega)} + \int_0^T ||(\bar{\rho} c)^2 u_b'||^2_{L^2(\partial\Omega_P)} dt \end{aligned} \tag{60}$$

From (57),

$$|q_b'|^2 \geq \gamma^2 \bar{p}^2 u_b'^2 = c^4 \bar{\rho}^2 u_b'^2 \tag{61}$$

so that

$$||q_b'||^2_{L^2(\partial\Omega_P)} = \int_{\partial\Omega_P} |q_b'|^2 dS \geq \int_{\partial\Omega_P} c^4 \bar{\rho}^2 u_b'^2 dS \geq \frac{1}{\max_{\partial\Omega_P}\{\bar{\rho}^2\}} ||(\bar{\rho} c)^2 u_b'||^2_{L^2(\partial\Omega_P)} \tag{62}$$

Substituting (62) into (60):

$$\begin{aligned} ||q'(\cdot,T)||^2_{(H^2,\Omega)} &\leq ||f(\cdot)||^2_{(H^2,\Omega)} + \max_{\partial\Omega_P}\{\bar{\rho}^2\} \int_0^T ||q_b'||^2_{L^2(\partial\Omega_P)} dt \\ &\leq K \left(||f(\cdot)||^2_{(H^2,\Omega)} + \int_0^T ||q_b'||^2_{L^2(\partial\Omega_P)} dt \right) \end{aligned} \tag{63}$$

Here, $K = \max\left\{ \max_{\partial\Omega_P}\{\bar{\rho}^2\}, 1 \right\}$. Referring to Definition 2.9 in [14] (restated in Section 9.9 of the Appendix), we see that (63) satisfies the definition of strong well-posedness, with $\alpha = 0$. Thus, the acoustically-reflecting boundary condition (47) is strongly well-posed under the assumptions of the claim. □

2.5 Stability of Acoustically-Reflecting Boundary Condition

Having established well-posedness of the acoustically-reflecting boundary condition, let us now study its stability. It turns out that the stability analysis is most illustrative when done in the original fluid variables q' rather than the characteristic variables V'. Write $q'_M \equiv \sum_{k=1}^{M} a_k \phi_k$, that is, q'_M is the numerical ROM solution. To study stability, consider the energy estimate $||q'_M||_H^2$. Let q'_b be the vector of plate boundary conditions on $\partial\Omega_P$, so that, to weakly enforce (47), one substitutes $q'_M \leftarrow q'_b$ on $\partial\Omega_P$. Then, neglecting for now the far-field boundary conditions (that is, assuming they have been imposed in a stable fashion) and assuming a uniform base flow,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||q'_M||_H^2 &= \frac{1}{2}\frac{d}{dt}(q'_M, q'_M)_{(H,\Omega)} \\
&= \int_\Omega q'^T_M H \frac{\partial q'_M}{\partial t} d\Omega \\
&= -\int_\Omega q'^T_M H A_i \frac{\partial q'_M}{\partial x_i} d\Omega \\
&= -\int_{\partial\Omega_P} q'^T_M H A_n q'_b dS + \int_\Omega q'^T_M H A_i \frac{\partial q'_M}{\partial x_i} d\Omega \\
&= -\int_{\partial\Omega_P} q'^T_M H A_n q'_b dS + \frac{1}{2}\int_\Omega \frac{\partial}{\partial x_i}\left(q'^T_M H A_i q'_M\right) d\Omega - \frac{1}{2}\int_\Omega q'^T_M \underbrace{\frac{\partial (HA_i)}{\partial x_i}}_{=0 \text{ (uniform base flow)}} q'_M d\Omega \\
&= -\int_{\partial\Omega_P} q'^T_M H A_n q'_b dS + \frac{1}{2}\int_{\partial\Omega_P} q'^T_M H A_n q'_M dS \\
&= \int_{\partial\Omega_P} q'^T_M H A_n \left(\frac{1}{2} q'_M - q'_b\right) dS
\end{aligned} \tag{64}$$

Note that when $q'_M = q'_b$ on $\partial\Omega_P$, (64) reduces to

$$\frac{1}{2}\frac{d}{dt}||q'_M||^2_{(H,\Omega)} = -\frac{1}{2}\int_{\partial\Omega_P} q'^T_b H A_n q'_b dS \tag{65}$$

which is precisely the expression that arises in the proof of Theorem 2.1.1; that is, the well-posedness condition (22) is recovered.

Let us now evaluate the integrand on the right-hand-side of (64). If the boundary condition is to be imposed in the characteristic variables $V'_b = S q'_b$, then the last line of (64) is

$$\frac{1}{2}\frac{d}{dt}||q'_M||^2_{(H,\Omega)} = \int_{\partial\Omega_P} \left(\frac{1}{2} q'^T_M H A_n q'_M - q'^T_M H S \Lambda V'_b\right) dS \tag{66}$$

with V'_b is as in (53). Assuming $\bar{u}_n \equiv 0$,

$$HS\Lambda V'_b = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}\bar{\rho}cn_1 & \frac{1}{2}\bar{\rho}cn_1 \\ 0 & 0 & 0 & \frac{1}{2}\bar{\rho}cn_2 & \frac{1}{2}\bar{\rho}cn_2 \\ 0 & 0 & 0 & \frac{1}{2}\bar{\rho}cn_3 & \frac{1}{2}\bar{\rho}cn_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} V'_1 \\ V'_2 \\ V'_3 \\ V'_4 \\ V'_4 - 2u'_b \end{pmatrix} = \begin{pmatrix} \bar{\rho}cn_1(V'_4 - u'_b) \\ \bar{\rho}cn_2(V'_4 - u'_b) \\ \bar{\rho}cn_3(V'_4 - u'_b) \\ 0 \\ u'_b \end{pmatrix} \tag{67}$$

so that, introducing the shorthand

$$u_{n,M} \equiv u_M n_1 + v_M n_2 + w_M n_3 \tag{68}$$

we have

$$q'^T_M H S \Lambda V'_b = \bar{\rho} c u'_{n,M}\left[u'_{n,M} - u'_b\right] + u'_{n,M} p'_M + u'_b p'_M \tag{69}$$

Since

$$HA_n q'_M = \begin{pmatrix} 0 & 0 & 0 & 0 & n_1 \\ 0 & 0 & 0 & 0 & n_2 \\ 0 & 0 & 0 & 0 & n_3 \\ 0 & 0 & 0 & 0 & 0 \\ n_1 & n_2 & n_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} u'_M \\ v'_M \\ w'_M \\ \zeta'_M \\ p'_M \end{pmatrix} = \begin{pmatrix} n_1 p'_M \\ n_2 p'_M \\ n_3 p'_M \\ 0 \\ u'_{n,M} \end{pmatrix} \tag{70}$$

one also has that

$$q'^T_M H A_n q'_M = 2 p'_M u'_{n,M} \tag{71}$$

It follows from (66) that

$$\frac{1}{2}\frac{d}{dt}||q'_M||^2_{(H,\Omega)} = \int_{\partial\Omega_P} \left[-\bar{\rho}cu'_{n,M}\left(u'_{n,M} - u'_b\right) - u'_b p'_M\right] dS = \int_{\partial\Omega_P} \left[-\bar{\rho}cu'^2_{n,M} + \left(\bar{\rho}cu'_{n,M} - p'_M\right)u'_b\right] dS \tag{72}$$

(72) gives rise to the following result.

Theorem 2.5.1. *Assume* $\bar{u}_n = 0$ *and* $\nabla\bar{q} = 0$. *Then both the linearized no-penetration plate boundary condition* (41) *and the acoustically-reflecting boundary condition* (47) *are stable for the fluid ROM. More specifically*

$$\frac{d}{dt}||q'_M||^2_{(H,\Omega)} \leq 0 \tag{73}$$

for the acoustically-reflecting condition (47) *and*

$$\frac{d}{dt}||q'_M||^2_{(H,\Omega)} = 0 \tag{74}$$

for the no-penetration condition (41)*.*

Proof. According to the definition of stability (see Section 9.10 in the Appendix), it is sufficient to consider the homogeneous version ($u'_b = 0$) of each boundary condition (41) and (47) to show stability of the more general inhomogeneous condition. For the acoustically-reflecting boundary condition, when $u'_b = 0$, the integrand on the right-hand-side of (72) reduces to

$$-\bar{\rho}cu'^2_{n,M} \leq 0 \tag{75}$$

which shows the first part of the claim.

For the linearized no-penetration boundary condition, recall from mechanics that the velocity of the plate is by definition the total derivative of the plate's displacement. In the case where the only component of the displacement vector that is non-zero is the z-component, this means that

$$u'_b = -\dot{\eta} - \bar{\mathbf{u}} \cdot \nabla\eta \tag{76}$$

When $u'_b \equiv 0$, the no-penetration condition (41) is thus

$$u'_n = u'_b = 0 \tag{77}$$

It follows that, substituting (77) into the fifth component of (70),

$$q'^T_M H A_n q'_b = q'^T_M \begin{pmatrix} n_1 p'_M \\ n_2 p'_M \\ n_3 p'_M \\ 0 \\ 0 \end{pmatrix} = u'_{n,M} p'_N \tag{78}$$

Then, making use of (71),

$$q'^T_M H A_n \left(\frac{1}{2}q'_M - q'_b\right) = \frac{1}{2}q'^T_M H A_n q'_M - q'^T_M H A_n q'_b = u'_{n,M} p'_M - u'_{n,M} = 0 \tag{79}$$

Thus, (74) holds. □

Theorem 2.5.1 shows that the no-penetration boundary condition (41) is neutrally-stable, that is, under this condition, the energy of the system remains constant in time.

Remark 6: Although the no-penetration boundary condition (41) is only neutrally-stable, meaning it lacks the energy "stability" margin of the acoustically-reflecting condition, it is nonetheless stable. One therefore cannot attribute the failure of the condition in enforcing $u'_n = 0$ at $\partial\Omega_P$ on the energy estimate (74). We emphasize that there is nothing wrong with the *condition* (41); it is the implementation of this condition (Algorithm 1) that is too weak. This explanation may be somewhat unsatisfying in light of footnote 15, which shows that the no-penetration condition (41) is mathematically equivalent to the acoustically-reflecting condition (47), and this latter condition is enforced weakly in exactly the same way as (41)(that is, following the procedure outlined in Algorithm 1). Whether there is a mathematical explanation for precisely why the weak implementation of one condition "works" and the other does not remains somewhat of an open question. One reason could be that (41) is a "momentum constraint", yet the weak implementation involving substituting $u'_n = u'_b$ into the boundary integral I_{P_j} in (46) only enforces "energy constraints".

3 Penalty-Type Solid Wall Boundary Treatment for the Fluid Equations

Up to this point, we have considered two possible conditions at the solid wall boundary $\partial\Omega_P$: the linearized no-penetration boundary condition (41) and the acoustically-reflecting condition (47). The usual way to weakly enforce boundary conditions in a numerical scheme is by applying them directly into the boundary integral, as in Algorithm 1. It has been argued, *cf.* [15], that this approach does not take into account the fact that the equation should be obeyed arbitrarily close to the boundary. To address this potential issue, a number of works, *cf.* [5], [10] and [15] have developed penalty and penalty-like enforcements of boundary conditions. Formulating a boundary condition using the penalty method amounts to rewriting a boundary value problem as:

$$\begin{cases} \mathscr{L}u - f = 0, & \text{in } \Omega \\ Bu = h, & \text{on } \partial\Omega \end{cases} \quad \rightarrow \quad \mathscr{L}u - f = -\Gamma(Bu - h)\delta_{\partial\Omega}, \quad \text{in } \Omega \cup \partial\Omega \tag{80}$$

Here, Γ is a diagonal matrix of penalty parameters selected such that stability is preserved and $\delta_{\partial\Omega}$ is an indicator function marking the boundary $\partial\Omega$.

In this section, we explore the penalty-formulated variant of the condition (47), focusing as before on well-posedness and stability.

3.1 Motivation

One motivation for considering a penalty enforcement of the boundary conditions for (3) is that a specific penalty formulation of the form $\frac{\partial q'}{\partial t} + A_i \frac{\partial q'}{\partial x_i} = -\Gamma(q' - q'_b)$ arises when one applies a boundary condition to the linearized Euler equations directly as done in (44) and "counter-integrates" by parts (line 3 of (81) below). Letting ϕ be a test function (or POD mode), we have that, denoting the vector of boundary data on $\partial\Omega_P$ by q'_b as before,

$$\begin{aligned} \left(\phi, \tfrac{\partial q'}{\partial t}\right)_{(H,\Omega)} &= -\textstyle\int_\Omega \phi^T H A_i \frac{\partial q'}{\partial x_i} d\Omega \\ &= -\textstyle\int_{\partial\Omega_P} \phi^T H A_n q'_b dS + \int_\Omega \frac{\partial \phi^T}{\partial x_i} H A_i q' d\Omega \\ &= \textstyle\int_{\partial\Omega_P} \phi^T H A_n q' dS - \int_{\partial\Omega_P} \phi^T H A_n q'_b dS - \int_\Omega \phi^T H A_i \frac{\partial q'}{\partial x_i} d\Omega \\ &= -\textstyle\int_{\partial\Omega_P} \phi^T H A_n (q'_b - q') dS - \int_\Omega \phi^T H A_i \frac{\partial q'}{\partial x_i} d\Omega \end{aligned} \tag{81}$$

so that

$$\left(\phi, \frac{\partial q'}{\partial t} + A_i \frac{\partial q'}{\partial x_i}\right)_{(H,\Omega)} = -\int_{\partial\Omega_P} \phi^T H A_n (q'_b - q') dS \tag{82}$$

(82) is the projection in the (H,Ω)-inner product of

$$\frac{\partial q'}{\partial t} + A_i \frac{\partial q'}{\partial x_i} = A_n (q' - q'_b) \delta_{\partial\Omega_P} \tag{83}$$

(83) is a *specific* penalty enforcement of the boundary condition $q' \leftarrow q'_b$ on $\partial\Omega_P$; that is, it has the form of the expression on the right of (80), with $-A_n$ playing the role of Γ.

Remark 7: Note that $-A_n$ is, in general, neither positive-definite nor diagonal, whereas we had defined Γ as a diagonal, positive definite matrix. For this reason, we say (83) is a penalty-*like* formulation. Actually, as we will show soon (Proposition 3.2.2) that if (83) is rewritten in the characteristic variables V', the penalty-like matrix that appears in this set of equations in place of $-A_n$ *is* diagonal and positive definite.

Let us take the analysis one step further. Assuming $\bar{u}_n \equiv 0$, for the acoustically-reflecting boundary condition (47), $HA_n(q'_b - q')$ evaluates to

$$HA_n(q'-q'_b) = \begin{pmatrix} n_1 p' \\ n_2 p' \\ n_3 p' \\ 0 \\ u'_n \end{pmatrix} - \begin{pmatrix} \bar{\rho} c n_1 \left(u'_n + \frac{\bar{\zeta}}{c} p' - u'_b\right) \\ \bar{\rho} c n_2 \left(u'_n + \frac{\bar{\zeta}}{c} p' - u'_b\right) \\ \bar{\rho} c n_3 \left(u'_n + \frac{\bar{\zeta}}{c} p' - u'_b\right) \\ 0 \\ u'_b \end{pmatrix} = \begin{pmatrix} -\bar{\rho} c n_1 (u'_n - u'_b) \\ -\bar{\rho} c n_2 (u'_n - u'_b) \\ -\bar{\rho} c n_3 (u'_n - u'_b) \\ 0 \\ u'_n - u'_b \end{pmatrix} \tag{84}$$

Then, taking $\phi = HA_n(q'-q'_b)$ in (82) and letting

$$\langle u, v\rangle_{(H^2,\partial\Omega)} \equiv \int_{\partial\Omega} u^T H^2 v dS, \qquad ||u||^2_{(H^2,\partial\Omega)} \equiv \langle u, u\rangle_{(H^2,\partial\Omega)} \tag{85}$$

one finds that

$$||A_n(q'-q'_b)||^2_{(H^2,\partial\Omega_P)} \quad = \int_{\partial\Omega_P} (1+\bar{\rho}^2 c^2)(u'_n - u'_b)^2 dS \tag{86}$$

where

$$||u'_n - u'_b||^2_{L^2(\partial\Omega_P)} \leq \int_{\partial\Omega_P} (1+\bar{\rho}^2 c^2)(u'_n - u'_b)^2 dS \leq \left(1 + \max_{\partial\Omega_P}\{\bar{\rho}^2 c^2\}\right) ||u'_n - u'_b||^2_{L^2(\partial\Omega_P)} \tag{87}$$

or

$$||u'_n - u'_b||^2_{L^2(\partial\Omega_P)} \leq ||A_n(q'-q'_b)||_{(H^2,\partial\Omega_P)} \leq \left(1 + \max_{\partial\Omega_P}\{\bar{\rho}^2 c^2\}\right) ||u'_n - u'_b||^2_{L^2(\partial\Omega_P)} \tag{88}$$

Here, $||\cdot||_{L^2(\partial\Omega)}$ is the usual L^2 norm over $\partial\Omega$. (88) relates the convergence of the vector q' to q'_b at the boundary $\partial\Omega_P$ to the convergence of u'_n to u'_b on $\partial\Omega_P$.

3.2 A Stable Penalty-like Formulation of the Acoustically-Reflecting Boundary Condition

As mentioned above, (83) resembles a specific penalty enforcement of the acoustically-reflecting boundary condition (47). Given (83), it is natural to ask what will happen if the A_n matrix on the right-hand-side of this equation is replaced by Γ, a diagonal matrix of penalty parameters. If one can derive a range of such parameters γ_i for which the enforcement of the boundary condition is stable, one will have a genuine penalty method for enforcing the condition.

Since (47) is specified in the characteristic variables, we will reformulate (83) in the characteristic variables:

$$\begin{array}{cc} \frac{\partial V'}{\partial t} + S^{-1}A_i S \frac{\partial V'}{\partial x_i} = -\Gamma[P^S V' - h^S]\delta_{\partial\Omega_P}, & \mathbf{x} \in \Omega \cup \partial\Omega_P, \quad 0 < t < T \\ V'(\mathbf{x},0) = f(\mathbf{x}), & \mathbf{x} \in \Omega \end{array} \tag{89}$$

Here P^S and h^S are given in (49) and $\Gamma = \text{diag}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ is a diagonal, positive definite matrix of penalty parameters. A sufficient condition for stability is that $\frac{d}{dt}||V'||^2_{(Q,\Omega)} \leq 0$ when $u'_b = 0$[17]. Setting $h^S \equiv 0$ and computing

[17] Again, by Definition 2.11 in [14], one need only consider the homogeneous boundary condition to show stability for general $u'_b \neq 0$; see Section 9.10 of the Appendix.

this energy estimate (omitting the first several steps, which are exactly the same as in (33)) gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||V'||^2_{(Q,\Omega)} &= \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega V'^T Q V' d\Omega \\ &= -\frac{1}{2}\int_{\partial\Omega_P} V'^T A_n^S V' dS - \int_{\partial\Omega_P} V'^T \Gamma Q P^S V' dS \\ &= \int_{\partial\Omega_P} V'^T \left[\left(-\frac{1}{2}A_n^S - \Gamma Q P^S\right)\right] V' dS \end{aligned} \tag{90}$$

Here,

$$A_n^S \equiv A_1^S n_1 + A_2^S n_2 + A_3^S n_3 = Q S^{-1} A_n S \tag{91}$$

The matrices $\{A_i^S = QS^{-1}A_i S : i = 1,2,3\}$ can be found in Section 9.8 of the Appendix.

From (90), $||V'||^2_{(Q,\Omega)}$ is non-increasing if the integrand in (90) is non-positive, that is if

$$V'^T \left[-\frac{1}{2}A_n^S - \Gamma Q P^S\right] V' \leq 0 \tag{92}$$

It is convenient to write (92) in matrix form as

$$V'^T \mathbf{H} V' \leq 0 \tag{93}$$

where

$$\mathbf{H} = -\frac{1}{2}A_n^S - \Gamma Q P^S = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & -\frac{c}{2} & \\ & & & \gamma_5 & -\gamma_5 + \frac{c}{2} \end{pmatrix} \tag{94}$$

(93) says that $\mathbf{H}$ must be negative semi-definite. Note that this matrix is not symmetric, since P^S is not symmetric. Let

$$\mathbf{H}^{symm} \equiv \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) = -\frac{1}{2}\left(A_n^S + \Gamma Q P^S + (P^S)^T Q \Gamma\right) \tag{95}$$

(the symmetric part of $\mathbf{H}$). Recall from linear algebra that a non-symmetric matrix is negative definite if and only if its symmetric part is negative definite (and likewise for semi-definiteness). Therefore to study stability, we will check the signs of the eigenvalues of $\mathbf{H}^{symm}$.

Theorem 3.2.1. *Assume $\bar{u_n} \equiv 0$ and $\nabla\bar{q} \equiv 0$. Then the penalty-enforced acoustically-reflecting boundary condition* (47) *is stable if*

$$\Gamma = cI_5 \tag{96}$$

where I_5 is the 5×5 identity matrix.

Proof. Due to the asymmetry of P^S and therefore $\mathbf{H}$, we must examine the eigenvalues of $\mathbf{H}^{symm} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T)$, given in (95). The eigenvalues of this matrix are:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0 \\ \lambda_3 &= 0 \\ \lambda_4 &= -\frac{1}{2}\gamma_5 + \frac{1}{2}\sqrt{2\gamma_5^2 - 2c\gamma_5 + c^2} \\ \lambda_5 &= -\frac{1}{2}\gamma_5 - \frac{1}{2}\sqrt{2\gamma_5^2 - 2c\gamma_5 + c^2} \end{aligned} \tag{97}$$

For $i = 1,2,3$, $\lambda_i = 0$, which clearly satisfies $\lambda_i \leq 0$. Also, $\lambda_5 \leq 0$ for all $\gamma_5 \geq 0$. Thus, the only eigenvalue that can be positive is λ_4. In fact, it is non-negative for all γ_5 but for stability, is it sufficient to require $\lambda_4 = 0$. Solving the equation $\lambda_4 = 0$ for γ_5 gives that $\gamma_5 = c$. Thus, a sufficient condition for stability of the penalty-enforced acoustically-reflecting boundary condition (47) is that, $\gamma_5 = c$, $\gamma_1, \cdots, \gamma_4 \geq 0$. To simplify the notation, we set $\gamma_1 = \cdots = \gamma_4 = c$, so that $\Gamma = cI_5$. □

Substituting the result of Theorem 3.2.1 into (89), we obtain the following penalty-like enforcement of the acoustically-reflecting boundary condition (47) in the characteristic variables:

$$\frac{\partial V'}{\partial t}+S^{-1}A_iS\frac{\partial V'}{\partial x_i}=-c(P^SV'-h^S)\delta_{\partial\Omega_P} \tag{98}$$

It turns out that (98) and the penalty-like formulation (83) that arose when the governing system of PDEs in the original variables was counter-integrated by parts are equivalent (Proposition 3.2.2). We emphasize that both are *specific* penalty formulations in which the "penalty parameter" on which stability depends is *fixed*. Although the penalty formulation presented here is motivated by classical penalty methods, the fact that stability is guaranteed only for a single value of the penalty parameter, rather than a range, distinguishes this approach from a "true" penalty method, in which the one typically sends $\gamma_i \to \infty$, reasoning that as γ_i gets large, the constraints (in this case, the boundary conditions) are better and better enforced.

Proposition 3.2.2. *Suppose $\bar{u}_n = 0$ and $\nabla\bar{q} = 0$. If the acoustically-reflecting boundary condition* (47) *is to be enforced on $\partial\Omega_P$, the stable penalty formulation* (98) *in the characteristic variables V' is equivalent to the penalty-like formulation* (83) *that arises when counter-integrating the linearized Euler equations in the original variables q' by parts.*

Proof. Rewritten in the characteristic variables, (83) is simply

$$\frac{\partial V'}{\partial t}+S^{-1}A_iS\frac{\partial V'}{\partial x_i}=\Lambda(V'-V'_b)_{\partial\Omega_P} \tag{99}$$

If the acoustically-reflecting boundary condition is applied on $\partial\Omega_P$, substituting V'_b given in (53),

$$\Lambda(V'-V'_b)=\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & c & \\ & & & & -c \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V'_5-V'_4+2u'_b \end{pmatrix}=-c\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V'_5-V'_4+2u'_b \end{pmatrix} \tag{100}$$

Now, turning to the left-hand-side of (98),

$$-c(P^SV'-h^S)=-c\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -V'_4+V'_5+2u'_b \end{pmatrix} \tag{101}$$

Comparing the right-hand-side of (100) with the right-hand-side of (101), we see that they are the same and that both are enforcing the acoustically-reflecting boundary condition $V'_5 = V'_4 - 2u'_b$. $\square$

Proposition 3.2.2 addresses the potential issue noted in Remark 7: while the penalty-like matrix $-A_n$ in (83) is neither diagonal nor positive definite, the matrix that plays the role of $-A_n$ when the penalty enforcement is done in the characteristic variables *is*. Indeed, one should not expect the penalty-like matrix in the original variables to be diagonal or positive definite since the boundary condition is being imposed in the characteristic variables.

4 Implementation of the Solid Wall Boundary Condition in the Fluid ROM

Having selected a boundary condition to be used at the solid wall boundary, namely the acoustically-reflecting boundary condition (47), let us now express this condition in terms of the ROM coefficients and basis functions. We do this by applying Algorithm 1: projecting the linearized Euler equations (3) onto a POD mode ϕ_j, integrating the spatial term by parts, and applying the boundary condition to the boundary integral that arises. From (44), the integral of interest is

$$I_P \equiv \int_{\partial\Omega_P} \phi_j^T H A_n q' dS = \int_{\partial\Omega_P} \phi_j^T H S \Lambda V' dS \tag{102}$$

Assume as we usual that $\bar{u}_n = 0$. From (67) and using the fact that $V_4' = u_n' + \frac{\bar{\zeta}}{c} p'$,

$$HS\Lambda V' = \begin{pmatrix} \bar{\rho} c n_1 (u_n' - u_b') + n_1 p' \\ \bar{\rho} c n_2 (u_n' - u_b') + n_2 p' \\ \bar{\rho} c n_3 (u_n' - u_b') + n_3 p' \\ 0 \\ u_b' \end{pmatrix} \tag{103}$$

so that, denoting

$$\phi_j^n \equiv \phi_j^1 n_1 + \phi_j^2 n_2 + \phi_j^3 n_3 \tag{104}$$

we have

$$\phi_j^T H S \Lambda V' = \bar{\rho} c \phi_j^n (u_n' - u_b') + p' \phi_j^n + u_b' \phi_j^5 \tag{105}$$

Inserting the modal representations of $q' = \sum_{k=1}^M a_k(t)\phi_k$ into (105) leads to the following term appearing in the j^{th} ROM equation:

$$I_{P_j} = \sum_{k=1}^M a_k(t) \left[\int_{\partial\Omega_P} \phi_j^n (\phi_k^5 + \bar{\rho} c \phi_k^n) dS \right] + \int_{\partial\Omega_P} (\phi_j^5 - \bar{\rho} c \phi_j^n) u_b' dS \tag{106}$$

(106) is the analog of (46) but for the acoustically-reflecting boundary condition (47). Expression (106) is different from (46), the expression arrived at in [17] for the no-penetration boundary condition (41) we had earlier. The following table compares the expressions that arise. Here, f_{P_j} is defined such that

$$I_{P_j} \equiv \int_{\partial\Omega_P} f_{P_j} dS \tag{107}$$

Solid Wall Boundary Condition	Expression for f_{P_j} in terms of ϕ_j and q'
Old no penetration BC (41)	$p'\phi_j^n + u_b'\phi_j^5$
New acoustically-reflecting BC (47)	$\bar{\rho} c \phi_j^n (u_n' - u_b') + p'\phi_j^n + u_b'\phi_j^5$

Note that the expression for f_{P_j} arising from the new acoustically-reflecting boundary condition is the same as the expression arising from the no-penetration boundary considered earlier except with an additional penalty-like term: $\bar{\rho} c (u_n' - u_b')$. As $u_n' \to u_b'$ on $\partial\Omega_P$, the new boundary condition converges to the old. Yet again, a penalty-like formulation hidden in the acoustically-reflecting boundary condition is revealed. Note that the "penalty" term $\bar{\rho}(u_n' - u_b')$ is multiplied by c, which is precisely the value of the penalty parameter derived in Theorem 3.2.1 to guarantee a stable enforcement of $V_5' = V_4' - 2u_b'$.

Remark 8: It is worth pointing out that (106) is not the only possible expression for the boundary integral I_{P_j} with the enforcement of the acoustically-reflecting boundary condition (47); it is the expression arising from a weak "IBP[18] + boundary integral substitution" implementation (Algorithm 1). The boundary condition can be implemented in other ways. For instance, one could start with equation (83) and project it onto the POD mode ϕ_j without doing any integrations by parts. Then the integrand f_{P_j} would be $\phi_j^T A_n (q' - q_b')$ (instead of $\phi_j^T H A_n q'$) which, when expressed in terms of the ROM coefficients and basis functions, would yield an expression different from (106). We emphasize that despite this difference, the same boundary condition, namely (47), is being enforced in both of these implementations. In other words, the total amount of information contained in the starting equations (3) is retained; the difference is in how it is distributed amongst the boundary and volume integrals that arise in the projection step.

[18] Integration by parts.

5 Coupled Fluid/Structure System

Recall from Sections 3 and 4 of [17] that the z-component of the displacement of the plate is governed by the following linearized von Karman equation:

$$(\rho_s h)\ddot{\eta} + D_{bend}(\nabla^4 \eta) = g \tag{108}$$

Here, g is the fluid pressure loading, ρ_s is the density of the plate material, h is the thickness of the plate, and D_{bend} is the bending stiffness[19].

Expanding the z-displacement η in its orthonormal, scalar ROM basis $\{\xi_k(x,y) : k = 1,2,\ldots,P\}$, substituting this expansion into (108), one arrives at the following set of ROM structure equations

$$(\rho_s h)\ddot{b}_k + \omega_k^2 b_k = G_k(t) \tag{109}$$

where

$$\omega_k^2 = D_{bend}(\nabla^4 \xi_k, \xi_k)_{L^2(\partial\Omega_P)} \tag{110}$$

$$G_k(t) = (g, \xi_k)_{L^2(\partial\Omega_P)} \tag{111}$$

$$g(x,y,t) = -p'(x,y,0,t) = -\sum_{k=1}^{M} a_k(t)\phi_k^5(x,y,0) \tag{112}$$

Everything on the structure side is exactly as derived in [17]. Denoting

$$S^T \equiv \begin{pmatrix} b_1(t) & \cdots & b_P(t) & \dot{b}_1(t) & \cdots & \dot{b}_P(t) \end{pmatrix} \in \mathbb{R}^{2P} \tag{113}$$

$$F^T \equiv \begin{pmatrix} a_1(t) & \cdots & a_M(t) \end{pmatrix} \in \mathbb{R}^{M} \tag{114}$$

(110) gives rise to the following matrix system:

$$\dot{S} = CF + DS \tag{115}$$

where

$$C \equiv \left(\begin{array}{ccc} & 0_{P\times M} & \\ \hline -\frac{1}{\rho_s h}\left(\phi_1^5, \xi_1\right)_{L^2(\partial\Omega_P)} & \cdots & -\frac{1}{\rho_s h}\left(\phi_M^5, \xi_1\right)_{L^2(\partial\Omega_P)} \\ \vdots & \ddots & \vdots \\ -\frac{1}{\rho_s h}\left(\phi_1^5, \xi_P\right)_{L^2(\partial\Omega_P)} & \cdots & -\frac{1}{\rho_s h}\left(\phi_M^5, \xi_P\right)_{L^2(\partial\Omega_P)} \end{array}\right) \equiv \left(\begin{array}{c} 0_{P\times M} \\ \hline -\frac{1}{\rho_s h}\tilde{C}_{P\times M} \end{array}\right) \tag{116}$$

$$D \equiv \left(\begin{array}{ccc|c} & 0_{P\times P} & & I_{P\times P} \\ \hline -\frac{D_{bend}}{\rho_s h}\left(\nabla^4 \xi_1, \xi_1\right)_{L^2(\partial\Omega_P)} & 0 & \cdots & \\ \vdots & \ddots & \vdots & 0_{P\times P} \\ 0 & \cdots & -\frac{D_{bend}}{\rho_s h}\left(\nabla^4 \xi_P, \xi_P\right)_{L^2(\partial\Omega_P)} & \end{array}\right) \equiv \left(\begin{array}{c|c} 0_{P\times P} & I_{P\times P} \\ \hline -\frac{1}{\rho_s h}\tilde{L}_{P\times P} & 0_{P\times P} \end{array}\right) \tag{117}$$

Similarly expanding the fluid equations in the orthonormal, vector fluid ROM basis $\{\phi_k(\mathbf{x}) : k = 1,2,\ldots,M\}$ yields the system

$$\dot{F} = AF + BS \tag{118}$$

The entries of the A and B matrices depend on the boundary conditions on $\partial\Omega_P$ and $\partial\Omega_F$. They are

$$A(i,j) = A_w(i,j) - \int_{\partial\Omega_F} h_j(\phi_i)dS + \int_{\Omega} \frac{\partial}{\partial x_i}(\phi_j^T H A_i) q' d\Omega, \qquad 1 \le i,j \le M \tag{119}$$

[19] Refer to Section 3.1 of [17] for the relation of D_{bend} to Young's modulus, Poisson's ratio, etc.

$$B(i,j) = \begin{cases} 0, & 1 \le i \le M, \;\; 1 \le j \le P \\ B_w(i,j), & 1 \le i \le M, \;\; (P+1) \le j \le 2P \end{cases} \tag{120}$$

with

	Old no-penetration BC (41)	New acoustically-reflecting BC (47)
$A_w(i,j)$	$-\int_{\partial\Omega_P} \phi_i^n \phi_j^5 dS$	$-\int_{\partial\Omega_P} \phi_i^n (\phi_j^5 + \bar{\rho} c \phi_j^n) dS$
$B_w(i,j)$	$\int_{\partial\Omega_P} \xi_{j-P} \phi_i^5 dS$	$\int_{\partial\Omega_P} \xi_{j-P} (\phi_i^5 - \bar{\rho} c \phi_i^n) dS$

and $h_j(\phi_i)$ determined by the far-field boundary conditions[20]. The coupled fluid/structure system is therefore

$$\begin{pmatrix} \dot{F} \\ \dot{S} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ S \end{pmatrix} \tag{121}$$

Here, B and C are the coupling matrices, which are also the matrices on which the stability of the coupled fluid/structure system depends.

5.1 Failure of Prior Energy Matrix Stability Analysis for Coupled System with New Acoustically-Reflecting Boundary Condition

Of particular interest is the stability of the coupled fluid/structure system (121). Recall that stability was shown under the old no-penetration boundary condition (41) assuming $\nabla \bar{q} \equiv 0$, $\bar{u}_n = 0$, $\bar{\mathbf{u}} = 0$ in [17] using energy matrices: energy matrices E_A and E_D for A and D respectively were exhibited such that $E_A B + (E_D C)^T = 0$; stability followed from Theorem 3.4 in [20][21]. Under the old condition (41), it was easy to define E_A and E_D such that $E_A B + (E_D C)^T = 0$, since the entries of B were negated multiples of the entries of C:

$$B_{\text{old}} = \left(\begin{array}{c|ccc} & (\phi_1^5, \xi_1)_{L^2(\partial\Omega_P)} & \cdots & (\phi_1^5, \xi_P)_{L^2(\partial\Omega_P)} \\ & \vdots & & \vdots \\ 0_{M\times P} & \vdots & \ddots & \vdots \\ & \vdots & & \vdots \\ & (\phi_M^5, \xi_1)_{L^2(\partial\Omega_P)} & \cdots & (\phi_M^5, \xi_P)_{L^2(\partial\Omega_P)} \end{array} \right) \equiv \left(\begin{array}{c|c} 0_{M\times P} & \tilde{C}^T_{M\times P} \end{array} \right) \tag{122}$$

$$C = \left(\begin{array}{ccc} & 0_{P\times M} & \\ \hline -\frac{1}{\rho_s h} (\phi_1^5, \xi_1)_{L^2(\partial\Omega_P)} & \cdots & -\frac{1}{\rho_s h} (\phi_M^5, \xi_1)_{L^2(\partial\Omega_P)} \\ \vdots & \ddots & \vdots \\ -\frac{1}{\rho_s h} (\phi_1^5, \xi_P)_{L^2(\partial\Omega_P)} & \cdots & -\frac{1}{\rho_s h} (\phi_M^5, \xi_P)_{L^2(\partial\Omega_P)} \end{array} \right) \equiv \left(\begin{array}{c} 0_{P\times M} \\ \hline -\frac{1}{\rho_s h} \tilde{C}_{P\times M} \end{array} \right) \tag{123}$$

The relatively simple choice of the diagonal matrices

$$E_A = I_{M\times M}, \qquad E_D = \left(\begin{array}{c|c} \tilde{L}_{P\times P} & 0_{P\times P} \\ \hline 0_{P\times P} & (\rho_s h) I_{P\times P} \end{array} \right) \tag{124}$$

"worked"; that is, the matrices E_A and E_D in (124) were energy matrices for A and D respectively and satisfied $E_A B + (E_D C)^T = 0$.

[20] See Section 2.3.13 in [17].

[21] Restated in Section 9.10.2 of the Appendix for convenience.

Unfortunately, things are not as simple for the new acoustically-reflecting boundary condition. Now,

$$
\begin{aligned}
B = B_{\text{new}} &= \left(\begin{array}{c|ccc} & -(\bar{\rho}c\phi_1^n, \xi_1)_{L^2(\partial\Omega_P)} + \left(\phi_1^5, \xi_1\right)_{L^2(\partial\Omega_P)} & \cdots & -(\bar{\rho}c\phi_1^n, \xi_P)_{L^2(\partial\Omega_P)} + \left(\phi_1^5, \xi_P\right)_{L^2(\partial\Omega_P)} \\ & \vdots & & \vdots \\ 0_{M\times P} & \vdots & \ddots & \vdots \\ & \vdots & & \vdots \\ & -(\bar{\rho}c\phi_M^n, \xi_1)_{L^2(\partial\Omega_P)} + \left(\phi_M^5, \xi_1\right)_{L^2(\partial\Omega_P)} & \cdots & -(\bar{\rho}c\phi_M^n, \xi_P)_{L^2(\partial\Omega_P)} + \left(\phi_M^5, \xi_P\right)_{L^2(\partial\Omega_P)} \end{array} \right) \\
&\equiv \left(\begin{array}{c|c} 0_{M\times P} & -\hat{C}^T_{M\times P} + \tilde{C}^T_{M\times P} \end{array} \right)
\end{aligned}
\tag{125}
$$

The C matrix remains the same (123) since the structure equations and fluid pressure loading are not altered. However, since B contains the additional $\hat{C}^T_{M\times P}$ submatrix whose components do not appear anywhere in the C matrix, defining the relevant energy matrices so as to apply Theorem 3.4 in [20] is rather difficult. It turns out that any matrices E_A and E_D satisfying $E_A B + (E_D C)^T = 0$ are *not* energy matrices for A and D respectively; in other words, it seems impossible to specify an energy matrix E_A for A and an energy matrix E_D for D such that $E_A B + (E_D C)^T = 0$ also holds. Note that Theorem 3.4 in [20] is a sufficient but *not* a necessary condition for stability. One therefore seeks an alternate analysis tool to attempt to try to prove stability of the new coupled system (121).

5.2 Stability of Structure Equations

Before studying the stability of the coupled system (121), one needs to make sure the fluid-only and structure-only systems ($\dot{F} = AF$ and $\dot{S} = CS$ respectively) are stable. Stability of the fluid equations under both condition (41) and (47) was shown in Section 3 (Theorem 2.5.1). For the sake of rigor, we formally prove stability of the structure system.

Theorem 5.2.1. *The von Karman equations governing the z-displacement of the plate* (10) *are stable.*

Proof. As before, it is sufficient to show stability for $g = 0$, which will imply stability for all $g \neq 0$ by Section 9.10 of the Appendix. Dividing both sides of (108) by $\rho_s h$ and setting $g = 0$, the z-displacement equation is

$$\ddot{\eta} + \frac{D_{bend}}{\rho_s h}(\nabla^4 \eta) = 0 \tag{126}$$

Let

$$\mathbf{r} \equiv \begin{pmatrix} \eta \\ \dot{\eta} \end{pmatrix}, \quad \mathbf{r}_P \equiv \sum_{k=1}^{P} \begin{pmatrix} b_k \\ \dot{b}_k \end{pmatrix} \xi_k \tag{127}$$

Then (126) can be written as

$$\dot{\mathbf{r}} + \underbrace{\begin{pmatrix} 0 & -1 \\ \frac{D_{bend}}{\rho_s h}\nabla^4 & 0 \end{pmatrix}}_{\equiv G} \mathbf{r} = 0 \tag{128}$$

or, substituting $\mathbf{r} \leftarrow \mathbf{r}_P$ and projecting onto ξ_k,

$$\begin{pmatrix} \dot{b}_k \\ \ddot{b}_k \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 \\ \frac{D_{bend}}{\rho_s h}(\xi_k, \nabla^4 \xi_k)_{L^2(\partial\Omega_P)} & 0 \end{pmatrix}}_{\equiv G_k} \begin{pmatrix} b_k \\ \dot{b}_k \end{pmatrix} = 0 \tag{129}$$

Now, the rate of change in energy of the solid-only system is

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\mathbf{r}_P||^2_{L^2(\partial\Omega_P)} &= \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega_P}\mathbf{r}_P^T\mathbf{r}_P dS \\
&= \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega_P}\left\{\sum_{k=1}^P\sum_{l=1}^P(\xi_k,\xi_l)_{L^2(\partial\Omega_P)}\begin{pmatrix} b_k & \dot{b}_k\end{pmatrix}\begin{pmatrix} b_l \\ \dot{b}_l\end{pmatrix}\right\}dS \\
&= \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega_P}\left\{\sum_{k=1}^P\sum_{l=1}^P\delta_{kl}\begin{pmatrix} b_k & \dot{b}_k\end{pmatrix}\begin{pmatrix} b_l \\ \dot{b}_l\end{pmatrix}\right\}dS \\
&= \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega_P}\left\{\sum_{k=1}^P\begin{pmatrix} b_k & \dot{b}_k\end{pmatrix}\begin{pmatrix} b_k \\ \dot{b}_k\end{pmatrix}\right\}dS \\
&= \int_{\partial\Omega_P}\sum_{k=1}^P\begin{pmatrix} b_k & \dot{b}_k\end{pmatrix}\begin{pmatrix} \dot{b}_k \\ \ddot{b}_k\end{pmatrix}dS \\
&= \sum_{k=1}^P\begin{pmatrix} b_k & \dot{b}_k\end{pmatrix}\underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{\omega_k^2}{\rho_s h} & 0\end{pmatrix}}_{-G_k}\begin{pmatrix} b_k \\ \dot{b}_k\end{pmatrix}dS
\end{aligned}
\tag{130}
$$

(using that $\omega_k^2 \equiv D_{bend}(\xi_k,\nabla^4\xi_k)_{L^2(\partial\Omega_P)}$). The Lyapunov condition for stability (see Section 9.10.3 in the Appendix) is that the real parts of the eigenvalues of the matrices $\{-G_k : k=1,2,\dots,P\}$ be non-positive. The eigenvalues of these matrices are $\pm\sqrt{-\frac{\omega_k^2}{\rho_s h}} = \pm\sqrt{\frac{\omega_k^2}{\rho_s h}}i$, since $\omega_k^2 \geq 0$ for all k, and $\rho_s, h > 0$ (recall that h is the thickness of the plate and ρ_s is the density of the plate material). Since the eigenvalues are all pure imaginary or 0, the Lyapunov condition holds, implying the last line of (130) is ≤ 0, as desired. It follows that the structure system is stable. $\square$

5.3 Stability of New Coupled System with Perturbed Fluid Pressure Loading

5.3.1 Possible Stability when $g = -p'$

The task of showing stability of the coupled fluid/structure system (121) under the new acoustically-reflecting boundary (47) condition turns out to be a challenging one. The application of classical methods for showing stability leads to an inconclusive result: the sufficient conditions for stability fail, meaning the system could be stable; but it could also be unstable. Recalling the definition of q'_M in Section 2.5 and $\mathbf{r}_P$ in (127), define the total energy of the coupled system as

$$
E \equiv \frac{1}{2}||q'_M||^2_{(H,\Omega)} + \frac{1}{2}||\mathbf{r}_P||^2_{L^2(\partial\Omega_P)} = \begin{pmatrix} q'^T_M & \mathbf{r}_P^T\end{pmatrix}\begin{pmatrix} \frac{1}{2}H & 0 \\ 0 & \frac{1}{2}I_2\delta_{\partial\Omega_P}\end{pmatrix}\begin{pmatrix} q'_M \\ \mathbf{r}_P\end{pmatrix}
\tag{131}
$$

Remark 9: (131) includes the coupling terms only if $u'_b, g \neq 0$. One should be careful in applying the definitions of stability in Section 9.10 of the Appendix to a coupled system such as (131). Naively setting u'_b and g to 0 per Definition 2.11 in [14] and bounding E would *not* show stability of the coupled system, since the coupling is contained precisely in u'_b and g.

First, suppose the function g is the pressure loading, so that $g = -p'_M$ on $\partial\Omega_P$. Since u'_b is the total derivative of the plate's displacement, in the case when the plate has a non-zero displacement only in the z-direction and $\bar{\mathbf{u}} \equiv 0$, one has that $\dot{\eta} = -u'_b$ so that $\mathbf{r}^T = \begin{pmatrix} \eta & -u'_b\end{pmatrix}$. Then, from the earlier analysis of the fluid and structure systems in isolation, (letting $\mathbf{e}_2^T \equiv \begin{pmatrix} 0 & 1\end{pmatrix}$)

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{1}{2}\frac{d}{dt}||q'_M||^2_{(H,\Omega)} + \frac{1}{2}\frac{d}{dt}||\mathbf{r}_P||^2_{L^2(\partial\Omega_P)} \\
&= \int_{\partial\Omega_P}\left[-\bar{\rho}cu'^2_{n,M} - \left(\bar{\rho}cu'_{n,M} - p'_M\right)\mathbf{r}_P^T\mathbf{e}_2\right]dS + \int_{\partial\Omega_P}\mathbf{r}_P^T(-G\mathbf{r}_P - p'_M\mathbf{e}_2)dS \\
&= \int_{\partial\Omega_P}\left[\mathbf{e}_2^T\left(-\bar{\rho}cu'^2_{n,M}\right)\mathbf{e}_2 - \mathbf{r}_P^T G\mathbf{r}_P - \mathbf{r}_P^T\bar{\rho}cu'_{n,M}\mathbf{e}_2\right]dS \\
&= \frac{d}{dt}(E_{\text{fluid only}}) + \frac{d}{dt}(E_{\text{structure only}}) + \int_{\partial\Omega_P}\bar{\rho}cu'_{n,M}u'_b dS
\end{aligned}
\tag{132}
$$

Remark 10: By Section 9.10 of the Appendix, a sufficient condition for stability is that $\frac{dE}{dt} \leq 0$. Actually, in the case of a coupled system such as (121), $\frac{dE}{dt} \leq 0$ is also a *necessary* condition for stability despite Definition 2.11 in [14] (see Section 9.10 of the Appendix). This is because the coupled fluid/structure equations describe a net, isolated *physical* system, whose energy cannot increase unless energy is being supplied from an outside source, which it is not.

(132) implies that if there is a "stability margin" in the fluid-only and/or structure-only systems (that is, if $\frac{d}{dt}(E_{\text{fluid only}}) < 0$ and/or $\frac{d}{dt}(E_{\text{structure only}}) < 0$), the coupled system can still be stable as long as

$$\int_{\partial\Omega_P} \bar{\rho} c u'_{n,M} u'_b dS \leq -\frac{d}{dt}(E_{\text{fluid only}}) - \frac{d}{dt}(E_{\text{structure only}}) \tag{133}$$

It was shown in Theorem 2.5.1 that there *is* in fact a stability margin in the fluid-only system under the new acoustically-reflecting boundary condition, a stability margin that was *not* available under the old no-penetration boundary condition. This observation suggests that the coupled fluid/structure system with $g = -p'$ *could* be stable, especially since one could prove stability for the coupled system arising from the application of the old no-penetration boundary condition despite the fact that it lacked a stability margin. Because one does not in general know the magnitude of the term on the left-hand-side of (133), however, one is unable to prove a general stability result for the new acoustically-reflecting boundary condition at this time without making additional assumptions.

5.3.2 Stability when $g = -p' + \mathscr{O}(u'_{n,M} - u'_b)$

It turns out that it *can* be shown that $\frac{dE}{dt} \leq 0$, which implies stability for the coupled fluid/structure system (121), if a perturbed fluid pressure loading

$$g = -p'_M + \mathscr{O}(u'_{n,M} - u'_b) \quad \text{on} \quad \partial\Omega_P \tag{134}$$

is assumed. This assumption is quite reasonable in practice: the $\mathscr{O}(u'_{n,M} - u'_b)$ term can be viewed as the numerical error. Indeed, due to finite precision arithmetic, even if one wishes to enforce $g = -p'_M$ on $\partial\Omega_P$, in implementations, one will only be able to enforce $g = -p'_M + \varepsilon$, where ε is some numerical or round-off error. Since one expects $u'_{n,M} \to u'_b$ on $\partial\Omega_P$, $g = -p'_M + \mathscr{O}(u'_{n,M} - u'_b) \approx -p'_M$, with $|g - (-p'_M)| \to 0$ as M, the number of POD snapshots, increases.

Remark 11: Just how quickly $u'_{n,M}$ converges to u'_b on $\partial\Omega_P$ is precisely the convergence rate of $||u'_{n,M} - u'_b||$ on $\partial\Omega_P$. The discovery that stability of the coupled fluid/structure system can be shown assuming a perturbed fluid pressure load thus leads naturally to an attempt to quantify the error $||u_{n,M} - u'_b||$, or more generally $||q' - q'_b||$, the topic of the next section.

Theorem 5.3.1. *Assume* $\bar{u}_n = 0$, $\nabla \bar{q} = 0$ *and we enforce the acoustically-reflecting boundary condition* (47) *on* $\partial\Omega_P$. *Suppose the fluid pressure loading is* $g = -p'_M + K(u'_{n,M} - u'_b)$ *on* $\partial\Omega_P$, *with* $K = -\bar{\rho}c$. *Then* $\frac{dE}{dt} \leq 0$ *(with* $\frac{dE}{dt}$ *defined in* (132)*), so that the coupled fluid/structure system* (121) *is stable.*

Proof. First, observe that

$$-\bar{\rho} c u'^2_{n,M} = -\bar{\rho} c (u'_{n,M} - u'_b)^2 - 2\bar{\rho} c u'_b (u'_{n,M} - u'_b) - \bar{\rho} c u'^2_b \tag{135}$$

With the new structure loading and using this relation, line 2 of (132) is

$$\begin{aligned}
\frac{dE}{dt} &= \int_{\partial\Omega_P} \left[-\bar{\rho} c (u'_{n,M} - u'_b)^2 - 2\bar{\rho} c u'_b (u'_{n,M} - u'_b) - \bar{\rho} c u'^2_b - \left(\bar{\rho} c u'_{n,M} - p'_M \right) \mathbf{r}_P^T \mathbf{e}_2 \right] dS \\
&\quad + \int_{\partial\Omega_P} \mathbf{r}_P^T (-G\mathbf{r}_P + [-p'_M + K(u'_{n,M} - u'_b)]\mathbf{e}_2) dS \\
&= \int_{\partial\Omega_P} \left[-\bar{\rho} c (u'_{n,M} - u'_b)^2 - 2\bar{\rho} c u'_b (u'_{n,M} - u'_b) - \bar{\rho} c u'^2_b - \bar{\rho} c u'_{n,M} u'_b - \mathbf{r}_P^T G \mathbf{r}_P - K(u'_{n,M} - u'_b) u'_b \right] dS \\
&= \int_{\partial\Omega_P} \left[-\mathbf{r}_P^T G \mathbf{r}_P - \bar{\rho} c (u'_{n,M} - u'_b)^2 - 2\bar{\rho} c u'_b (u'_{n,M} - u'_b) + \bar{\rho} c u'_b (u'_{n,M} - u'_b) - K(u'_{n,M} - u'_b) u'_b \right] dS \\
&= \int_{\partial\Omega_P} \left[-\mathbf{r}_P^T G \mathbf{r}_P - \bar{\rho} c (u'_{n,M} - u'_b)^2 - \bar{\rho} c u'_b (u'_{n,M} - u'_b) - K(u'_{n,M} - u'_b) u'_b \right] dS
\end{aligned} \tag{136}$$

If $K = -\bar{\rho}c$, the $u'_b(u'_{n,N} - u'_b)$ terms cancel. Then

$$\begin{aligned} \frac{dE}{dt} &= \int_{\partial\Omega_P} \left[-\mathbf{r}_P^T G \mathbf{r}_P - \bar{\rho}c(u'_{n,M} - u'_b)^2 \right] dS \\ &= \frac{d}{dt}(E_{\text{structure only}}) - \int_{\partial\Omega_P} \bar{\rho}c(u'_{n,M} - u'_b)^2 dS \\ &\leq 0 \end{aligned} \tag{137}$$

provided the structure-only system is stable, which it is by Theorem 5.2.1. □

6 Error Estimation and Convergence Analysis

Error quantification and convergence analysis of Reduced Order Models has yet to be placed on firm mathematical footing. Some attempts have been made in [6], [18] and [22]. One difficulty in quantifying the error in a ROM is that the span of the POD basis is not complete in $\mathscr{H}(\Omega)$, the Hilbert space to which the exact solution belongs. It is only complete in an *average* sense: since the POD basis contains only information of the kinematics of the flow field that were already encoded in the observations, it cannot be expected to contain all the features present in the exact analytical solution. Given the fact that a ROM is derived from another numerical solution, namely the full CFD solution, it is most natural to define the error in the ROM as the difference between the ROM solution and the CFD solution. One may then try to bound this error as a function of M, the number of POD modes retained in the ROM.

Remark 12: Note that the POD/Galerkin approach used in constructing the ROM discussed herein differs from classical POD/Galerkin methods. In most reduced order models that utilize the POD/Galerkin approach, the *discretized* equations are projected onto the POD modes. Our approach consists of two steps: calculation of a reduced basis using the POD of an ensemble of flow field realizations, followed by a Galerkin projection of the governing system of PDEs onto the reduced basis (à la (43)). In particular, we project the *continuous* linearized Euler equations (3) onto the POD modes, substituting the POD expansions into the arising integrals (Algorithm 1 and Section 4)[22]. One should therefore be careful in applying error estimates derived in other works, e.g., [22], as the derivations do not carry over directly due to this fundamental difference in the projection step of the ROM.

In this section, we derive bounds for the error in the ROM solution, $||q'_M - q'||_{(H,\Omega)}$, adapting procedures presented in [10], [18] and [22]. These estimates show that the ROM solution will not blow up in finite time, which is yet another stability result.

6.1 Mathematical Preliminaries

In the upcoming error analysis, the following three solutions, belonging to the following three spaces, are of interest:

$$\begin{array}{ll} \text{Exact solution to (151)}: & q'(\mathbf{x},t) \in \mathscr{V} \subset \mathbb{R}^5 \\ \text{Computed CFD solution}: & q'_h(\mathbf{x},t) \in \mathscr{V}^h \subset \mathbb{R}^5 \\ \text{Computed ROM solution}: & q'_M(\mathbf{x},t) \in \mathscr{V}^M \subset V^h \subset \mathbb{R}^5 \end{array} \tag{138}$$

Here, $\mathscr{V}^M \subset \mathscr{V}^h \subset \mathscr{V} \subset \mathbb{R}^5$ are vector spaces. Defining an inner product on each of these spaces turns the space into a Hilbert space. One can define more than one inner product on these spaces, and it turns out that two inner products

[22]There are two main reasons for projecting the continuous equations to build a ROM: doing so enables one to construct a stable ROM for any approximation basis, and the ROM-building machinery can be implemented independent of the CFD simulation code, a more non-intrusive approach. Refer to [2], [3] and [4].

are of particular interest to us: for $u, v \in \mathscr{V}$,

$$(u,v)_{(H,\Omega)} \equiv \int_\Omega u^T H v d\Omega \tag{139}$$

$$\left((u,v)^{avg}_{(H,\Omega)}\right)^2 \equiv \langle (u,v)_{(H,\Omega)} \rangle \equiv \frac{1}{T}\int_0^T (u,v)^2_{(H,\Omega)} dt \tag{140}$$

(140) is a continuous time-average of (140) (averaging being denoted by $\langle \cdot \rangle$). Each inner product induces a norm: $||v||^2 \equiv (v,v)$. It was shown in [2] that the norm induced by (139) is indeed a valid norm, H being positive definite. For the sake of rigor, let us prove that the inner product (140) also induces a valid norm.

Lemma 6.1.1. *Let $v \in \mathscr{V}$. Then the inner product* (140) *induces the so-called time-averaged (H,Ω)-norm, given by*

$$||v||^{avg}_{(H,\Omega)} = \sqrt{\frac{1}{T}\int_0^T ||v||^2_{(H,\Omega)} dt} \tag{141}$$

(141) *defines a valid norm on $\mathscr{V}$, turning the space into the Hilbert space denoted by $\mathscr{H}_{avg}(\Omega)$.*

Proof. To show that (141) defines a norm, we check the following norm axioms (homogeneity, positive definiteness and triangle inequality). We make use of the fact that $||\cdot||_{(H,\Omega)}$ is known to be a norm, and hence satisfies all three norm axioms.

1. Homogeneity: let $a \in \mathbb{R}$. Then

   $$\left(||av||^{avg}_{(H,\Omega)}\right)^2 = \int_0^T ||av||^2_{(H,\Omega)} dt = a^2 \frac{1}{T}\int_0^T ||v||^2_{(H,\Omega)} dt = a^2 \left(||v||^{avg}_{(H,\Omega)}\right)^2 \tag{142}$$

   from which it follows that $||av||^{avg}_{(H,\Omega)} = |a| ||v||^{avg}_{(H,\Omega)}$.

2. Positive definiteness: that is, we would like to show that $||v||^{avg}_{(H,\Omega)} \geq 0$ with $||v||^{avg}_{(H,\Omega)} = 0$ if and only if $v \equiv 0$. This clearly follows from the positive definiteness of $||\cdot||_{(H,\Omega)}$.

3. Triangle inequality: Let $v, u \in \mathscr{V}$. Then

   $$\begin{aligned} ||v+u||^{avg}_{(H,\Omega)} &= \left(\tfrac{1}{T}\int_0^T ||v+u||^2_{(H,\Omega)} dt\right)^{1/2} \\ &\leq \left(\tfrac{1}{T}\int_0^T \left(||v||_{(H,\Omega)} + ||u||_{(H,\Omega)}\right)^2 dt\right)^{1/2} \\ &\leq \left(\tfrac{1}{T}\int_0^T ||v||^2_{(H,\Omega)} dt\right)^{1/2} + \left(\tfrac{1}{T}\int_0^T ||u||^2_{(H,\Omega)} dt\right)^{1/2} \\ &= ||v||^{avg}_{(H,\Omega)} + ||u||^{avg}_{(H,\Omega)} \end{aligned} \tag{143}$$

   To go from the second to the third line of (143), one applies the Minkowski inequality with $p = 2$ (see Section 9.3 in the Appendix).

Since (141) satisfies the three norm axioms, it is indeed a norm on $\mathscr{V}$. □

We will call $||\cdot||_{(H,\Omega)}$ the "(H,Ω)-norm", and $||\cdot||^{avg}_{(H,\Omega)}$ the "time-averaged (H,Ω)-norm"[23].

Let us now define the following infinite-dimensional Hilbert spaces, obtained by specifying an inner product on $\mathscr{V}$:

Hilbert Space	Vector Space	+	Inner Product
$\mathscr{H}(\Omega)$	$\mathscr{V}$		$(\cdot,\cdot)_{(H,\Omega)}$
$\mathscr{H}_{avg}(\Omega)$	$\mathscr{V}$		$(\cdot,\cdot)^{avg}_{(H,\Omega)}$

[23]Note that $||\cdot||^{avg}_{(H,\Omega)}$ as defined in (141) is a *continuous* time-average. In reality, one is likely to have the discrete analog of averaged norm; see Remark 13.

and similarly for the subspaces $\mathscr{V}^h$ and $\mathscr{V}^M$ (that is, for example, $\mathscr{H}^h(\Omega)$ is the Hilbert space defined by equipping the vector space $\mathscr{V}^h$ with the inner product $(\cdot,\cdot)_{(H,\Omega)}$; $\mathscr{H}^h_{avg}(\Omega)$ is the Hilbert space defined by equipping the vector space $\mathscr{V}^h$ with the inner product $(\cdot,\cdot)^{avg}_{(H,\Omega)}$).

6.1.1 Proper Orthogonal Decomposition (POD) and the Method of Snapshots

The Proper Orthogonal Decomposition (POD) is a mathematical procedure that, given an ensemble of data, constructs a basis for that ensemble that is optimal in a well-defined sense[24]. Let $\{\phi_i \in \mathscr{V}^h : i = 1,2,\dots,N\}$ be a basis for $\mathscr{V}^h$ (assuming dim$\mathscr{V}^h = N$). POD seeks an M-dimensional ($M << N$) subspace $\mathscr{V}^h$ spanned by the set $\{\phi_i \in \mathscr{V}^h : i = 1,2,\dots,M\}$ such that the total square distance between q'_h and its orthogonal projection onto $\mathscr{V}^M$ is minimized; that is, it seeks the set $\{\phi_i\}$ solves the following constrained optimization problem over $\mathscr{H}_{avg}(\Omega)$:

$$\begin{array}{cc} \min_{\{\phi_i\}_{i=1}^M} & \left(||q'_h - \Pi_M q'_h||^{avg}_{(H,\Omega)}\right)^2 \\ \text{subject to} & (\phi_i,\phi_j)_{(H,\Omega)} = \delta_{ij},\ 1 \le i \le M, 1 \le j \le i \end{array} \tag{144}$$

Here, $\Pi_M : \mathbb{R}^5 \to \mathscr{V}^M$ is an orthogonal projection operator[25] onto the subspace $\mathscr{V}^M$. By definition, Π_M has the following properties:

1. For all $u \in \mathscr{V}$, $\Pi_M(\Pi_M u) = \Pi_M u$ [that is, Π_M is *idempotent*].
2. For all $u,v \in \mathscr{V}$, $\Pi_M(u+v) = \Pi_M u + \Pi_M v$ [that is, Π_M is *linear*].
3. $||\Pi_M|| = 1$ for any norm $||\cdot||$ on $\mathscr{V}$ [a consequence 1. above].
4. For all $u \in \mathscr{V}$, $\frac{\partial(\Pi_M u)}{\partial t} = \Pi_M\left(\frac{\partial u}{\partial t}\right)$ [that is, Π_M is a spatial-only operator, so time-differentiation commutes with projection].
5. For all $v \in \mathscr{V}^M$, $\Pi_M v = v$.
6. For all $v \in (\mathscr{V}^M)^\perp$, $\Pi_M v = 0$ [here $(\mathscr{V}^M)^\perp$ denotes the subspace orthogonal to $\mathscr{V}^M$].

It is a well-known result (*cf.* [2], [16], [18] and [22]) that the solution to (144) reduces to an eigenvalue problem:

$$\mathscr{R}\phi = \lambda\phi \tag{145}$$

where

$$\mathscr{R}\phi = \langle q'_h(q'_h,\phi)_{(H,\Omega)}\rangle \tag{146}$$

The operator $\mathscr{R}$ is self-adjoint and non-negative definite. If one further assumes that $\mathscr{R}$ is compact, then there exists a countable set of non-negative eigenvalues λ_i with associated eigenfunctions ϕ_i. These eigenfunctions form an orthonormal subspace of $\mathscr{H}_{avg}(\Omega)$, namely $\mathscr{H}^M_{avg}(\Omega)$. In the context of the ROM, the natural definition of the projection operator $\Pi_M : \mathbb{R}^5 \to \mathscr{V}^M$ is: for $q' \in \mathscr{V}$

$$\Pi_M q' = \sum_{k=1}^{M} \left(\phi_k, q'\right)_{(H,\Omega)} \phi_k \tag{147}$$

Letting $\lambda_1 \le \cdots \le \lambda_M \le \cdots \le \lambda_N$ be the ordered eigenvalues of $\mathscr{R}$, the minimum value of the objective function in (144) over all M dimensional subspaces $\mathscr{V}^M$ is $\sum_{j=M+1}^{N} \lambda_j$, that is, as $N \to \infty$

$$||q'_h - \Pi_M q'_h||^{avg}_{(H,\Omega)} = \sqrt{\sum_{j=M+1}^{N} \lambda_j} \tag{148}$$

[24]Refer to Chapter 3 of [16] for an in depth overview of POD.
[25]Note that Π_M can project from either of the spaces $\mathscr{V}$ or $\mathscr{V}^h$; we therefore write the domain as $\mathbb{R}^5$, as this larger space contains both $\mathscr{V}$ and $\mathscr{V}^h$.

The set of M eigenfunctions $\{\phi_i : i = 1, 2, \ldots, M\}$ corresponding to the M largest eigenvalues of $\mathscr{R}$ is precisely the set of $\{\phi_i\}$ that solves (144). Note that it is constrained to be orthonormal in the (H,Ω)-norm. As mentioned at the beginning of Section 6, we emphasize that the POD basis is *not* complete in $\mathscr{H}_{avg}(\Omega)$. It is, however, complete in the sense that, on average, any snapshot used to construct it can be represented, that is, $\left|\left|q' - \sum_j (q', \phi_j)_{(H,\Omega)}\phi_j\right|\right|^{avg}_{(H,\Omega)} = 0$.

Remark 13: Note that in the derivations presented herein, we have assumed that the norm on $\mathscr{H}_{avg}(\Omega)$ is computed as a *continuous* time average (141). In actual ROM computations, one will have a discrete analog of this continuous norm:

$$||v||^{avg}_{(H,\Omega)} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} ||v(\cdot,t_i)||^2_{(H,\Omega)}} = \sqrt{\sum_{j=M+1}^{N} \lambda_j} \tag{149}$$

where N is the total number of snapshots. Since

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} ||v(\cdot,t_i)||^2_{(H,\Omega)} = \frac{1}{T}\int_0^T ||v(\cdot,t)||^2_{(H,\Omega)} dt \tag{150}$$

technically the results of Section 6.3 below are technically valid in the limit as $N \to \infty$; see also Remark 15.

In preparation for the upcoming convergence analysis, let us summarize the key equations that each of the solutions q', q'_h and q'_M in (138) are assumed to satisfy.

6.1.2 The Exact Solution q'

The exact solution $q' \in \mathscr{V}$ is the solution satisfying

$$\begin{aligned} \underbrace{\frac{\partial q'}{\partial t} + A_i\frac{\partial q'}{\partial x_i}}_{\equiv \mathscr{L}q'} &= 0, \quad \mathbf{x} \in \Omega, \qquad 0 < t < T \\ q' - q'_b &= 0, \qquad \mathbf{x} \in \partial\Omega_P, \quad 0 < t < T \\ q'(\mathbf{x},0) &= f(\mathbf{x}), \quad\ \mathbf{x} \in \Omega \end{aligned} \tag{151}$$

Here $f : \Omega \to \mathbb{R}$ is a given function and $q'_b = SV'_b$, where V'_b is the vector defining the plate boundary condition in the characteristic variables (see (53) and (55) for V'_b and q'_b respectively for the acoustically-reflecting boundary condition (47) considered here). $\mathscr{L}$ is a linear, spatial differential operator.

6.1.3 The CFD Solution q'_h

The CFD solution $q'_h(\mathbf{x},t) \in \mathscr{V}^h \subset \mathbb{R}^5$ in (138) is piecewise continuous in space and in time. In the numerical implementation, the CFD solution q'_h will actually be semi-discrete: it is discrete in space[26] and continuous in time. Discretizing the domain Ω into n grid-points and denoting the CFD solution at the i^{th} grid-point as $q'_h(x_i,t)$, the CFD solution vector (containing values of the solution at each of the n grid-points) at time t is then

$$\mathbf{q}'_h(t) \equiv \begin{pmatrix} q'_h(x_1,t) \\ \vdots \\ q'_h(x_n,t) \end{pmatrix} \tag{152}$$

In the current implementation, the CFD data are represented as piecewise linear fields and the vector (152) belongs to the finite element space of linear tetrahedral elements. $\mathbf{q}_h(t)$ satisfies a linear dynamical system of the form

$$\dot{\mathbf{q}}_h{}' = A\mathbf{q}'_h + u \tag{153}$$

[26] Discretized by a finite element representation.

where A is a $5n \times 5n$ matrix and u is a $5n$-vector.

6.1.4 The ROM Solution q'_M

The analysis in Section 6.2, motivated primarily by [10], requires an equation for $q'_M \in \mathscr{V}^M$, the computed ROM solution. We will say that q'_M satisfies the following IBVP with a penalty-type correction at the plate boundary:

$$\begin{array}{ll} \frac{\partial q'_M}{\partial t} + A_i \frac{\partial q'_M}{\partial x_i} = -\Gamma[q'_M - q'_b]\delta_{\partial\Omega_P}, & \mathbf{x} \in \Omega \cup \partial\Omega_P, \quad 0 < t < T \\ q'_M(\mathbf{x}, 0) = f(\mathbf{x}), & \mathbf{x} \in \Omega \end{array} \tag{154}$$

Here, Γ is a penalty-like matrix specified such that (154) is stable. Recall that this matrix Γ was determined in Section 3.1 to be $-A_n$. In the subsequent analysis, we will make use of this result, setting $\Gamma = -A_n$ in (154).

6.2 Error Estimates in the Hilbert Space $\mathscr{H}(\Omega)$

We first bound the error in the (H,Ω)-norm, that is, viewing the solution q' as belonging to the Hilbert space $\mathscr{H}(\Omega)$. Our ultimate goal is to relate the error in the ROM solution, $||q' - q'_M||_{(H,\Omega)}$, to bounded quantities such as $||q'_h - \Pi_M q'_h||_{(H,\Omega)}$ and $||q' - q'_h||_{(H,\Omega)}$ for which one can obtain some kind of numerical estimates.

The upshot to selecting the (H,Ω)-norm over the time-averaged (H,Ω)-norm is that the resulting error bound is valid for any time $t \in [0,T]$ rather than in an average sense. The downside is that the quantity $||q'_h - \Pi_M q'_h||_{(H,\Omega)}$ is unknown, whereas $||q'_h - \Pi_M q'_h||^{avg}_{(H,\Omega)}$ is given by (148) above. In Section 6.3, we will derive the same error bound, except in the other Hilbert space $\mathscr{H}_{avg}(\Omega)$ using the time-averaged (H,Ω)-norm so as to make use of (148).

Let $q' \in \mathscr{V}$ and $q'_M \in \mathscr{V}^M$. Denote $E \equiv \Pi_M q' - q'_M$, where $\Pi_M : \mathbb{R}^5 \to \mathscr{V}^M$ is an orthogonal projection operator satisfying properties 1-6 listed in Section 6.1.1. Applying Π_M to (151) gives

$$\begin{array}{ll} \frac{\partial(\Pi_M q')}{\partial t} + A_i \frac{\partial(\Pi_M q')}{\partial x_i} + \left[\Pi_M\left(A_i \frac{\partial q'}{\partial x_i}\right) - A_i \frac{\partial(\Pi_M q')}{\partial x_i}\right] = 0, & \mathbf{x} \in \Omega, \qquad 0 < t < T \\ \Pi_M(q' - q'_b) = 0, & \mathbf{x} \in \partial\Omega_P, \quad 0 < t < T \\ \Pi_M q'(\mathbf{x}, 0) = \Pi_M f(\mathbf{x}), & \mathbf{x} \in \Omega \end{array} \tag{155}$$

Now, subtracting (154) from (155), one has that

$$\begin{array}{ll} \frac{\partial E}{\partial t} + A_i \frac{\partial E}{\partial x_i} + W_1 = A_n[E - E_b]\delta_{\partial\Omega_P}, & \mathbf{x} \in \Omega \cup \partial\Omega_P, \quad 0 < t < T \\ E(\mathbf{x}, 0) = \Pi_M f(\mathbf{x}) - f(\mathbf{x}), & \mathbf{x} \in \Omega \end{array} \tag{156}$$

where $E_b \equiv \Pi_M q'_b - q'_b$ and

$$W \equiv \Pi_M\left(A_i \frac{\partial q'}{\partial x_i}\right) - A_i \frac{\partial(\Pi_M q')}{\partial x_i} \tag{157}$$

Using the shorthand defined in (16) and applying the integration by parts "trick"[27] with uniform base flow to go from

[27] See section 9.1 of the Appendix.

line 4 to line 5,

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}||E||^2_{(H,\Omega)} &= \tfrac{1}{2}\tfrac{\partial}{\partial t}(E,E)_{(H,\Omega)} \\
&= (E_t,E)_{(H,\Omega)} \\
&= -\left(A_i\tfrac{\partial E}{\partial x_i}+W,E\right)_{(H,\Omega)} + \textstyle\int_{\partial\Omega_P} E^T HA_n E dS - \int_{\partial\Omega_P} E^T HA_n E_b dS \\
&= -\textstyle\int_\Omega E^T HA_i \tfrac{\partial E}{\partial x_i} d\Omega - (W,E)_{(H,\Omega)} + \int_{\partial\Omega_P} E^T HA_n E dS - \int_{\partial\Omega_P} E^T HA_n E_b dS \\
&= -\tfrac{1}{2}\textstyle\int_\Omega \tfrac{\partial(E^T HA_i E)}{\partial x_i} d\Omega - (W,E)_{(H,\Omega)} + \int_{\partial\Omega_P} E^T HA_n E dS - \int_{\partial\Omega_P} E^T HA_n E_b dS \\
&= -\tfrac{1}{2}\textstyle\int_{\partial\Omega_P} E^T HA_n E dS - (W,E)_{(H,\Omega)} + \int_{\partial\Omega_P} E^T HA_n E dS - \int_{\partial\Omega_P} E^T HA_n E_b dS \\
&= \textstyle\int_{\partial\Omega_P} E^T HA_n\left(\tfrac{1}{2}E - E_b\right) dS - (W,E)_{(H,\Omega)}
\end{aligned} \tag{158}$$

In order to proceed, let us examine further the first term in the last line of (158). Expanding out this integral using the definitions $E \equiv \Pi_M q' - q'_M$ and $E_b \equiv \Pi_M q'_b - q'_b$,

$$\begin{aligned}
\textstyle\int_{\partial\Omega_P} E^T HA_n\left(\tfrac{1}{2}E - E_b\right) dS &= \textstyle\int_{\partial\Omega_P} (\Pi_M q' - q'_M)^T HA_n\left(\tfrac{1}{2}\Pi_M q' - \tfrac{1}{2}q'_M - \Pi_M q'_b + q'_b\right) dS \\
&= \textstyle\int_{\partial\Omega_P} \left\{\tfrac{1}{2}(\Pi_M q')^T HA_n \Pi_M q' - \tfrac{1}{2}(\Pi_M q')^T HA_n q'_M - (\Pi_M q')^T HA_n (\Pi_M q'_b) + (\Pi_M q')^T HA_n q'_b \right. \\
&\quad \left. -\tfrac{1}{2}(q'_M)^T HA_n \Pi_M q' + \tfrac{1}{2}(q'_M)^T HA_n q'_M + (q'_M)^T HA_n \Pi_M q'_b - (q'_M)^T HA_n q'_b\right\} dS \\
&= \textstyle\int_{\partial\Omega_P} \left\{\tfrac{1}{2}(\Pi_M q')^T HA_n \Pi_M q' - (\Pi_M q')^T HA_n q'_M - (\Pi_M q')^T HA_n (\Pi_M q'_b) + (\Pi_M q')^T HA_n q'_b \right. \\
&\quad \left. +\tfrac{1}{2}(q'_M)^T HA_n q'_M + (q'_M)^T HA_n \Pi_M q'_b - (q'_M)^T HA_n q'_b\right\} dS
\end{aligned} \tag{159}$$

Denote

$$q'_M \equiv \begin{pmatrix} u'_M \\ v'_M \\ w'_M \\ \zeta'_M \\ p'_M \end{pmatrix}, \quad \Pi_M q' \equiv \begin{pmatrix} u'_\Pi \\ v'_\Pi \\ w'_\Pi \\ \zeta'_\Pi \\ p'_\Pi \end{pmatrix} \tag{160}$$

Recall from (55), (67) and (70) that

$$q'_b = \begin{pmatrix} cn_1(u'_{n,M} - u'_b) + n_1 p'_M \\ cn_2(u'_{n,M} - u'_b) + n_1 p'_M \\ cn_3(u'_{n,M} - u'_b) + n_1 p'_M \\ -\bar{\zeta} u'_b \\ \gamma\bar{p} u'_b \end{pmatrix}, \quad HA_n q'_b = \begin{pmatrix} \bar{\rho} cn_1(u'_{n,M} - u'_b) + n_1 p'_M \\ \bar{\rho} cn_2(u'_{n,M} - u'_b) + n_2 p'_M \\ \bar{\rho} cn_3(u'_{n,M} - u'_b) + n_3 p'_M \\ 0 \\ u'_b \end{pmatrix}, \quad HA_n q'_M = \begin{pmatrix} n_1 p'_M \\ n_2 p'_M \\ n_3 p'_M \\ 0 \\ u'_{n,M} \end{pmatrix} \tag{161}$$

Since Π_M is linear (by property 2 in Section 6.1.1), for any index $i = 1,\ldots,5$, $[\Pi_M q'_b](i) = [q'_b(i)]_\Pi$ (here $q'(i)$ denotes the i^{th} component of q'). Then

$$\frac{1}{2}(\Pi_M q')^T HA_n \Pi_M q' = u'_{n,\Pi} p'_\Pi \tag{162}$$

$$-(\Pi_M q')^T HA_n q'_M = -u'_{n,\Pi} p'_M - p'_\Pi u'_{n,M} \tag{163}$$

$$-(\Pi_M q')^T HA_n (\Pi_M q'_b) = -\bar{\rho} c(u'_{n,\Pi} - u'_b) u'_{n,\Pi} - u'_{n,\Pi} p'_\Pi - p'_\Pi u'_b \tag{164}$$

$$(\Pi_M q')^T HA_n q'_b = \bar{\rho} c(u'_{n,M} - u'_b) u'_{n,\Pi} + u'_{n,\Pi} p'_M + p'_\Pi u'_b \tag{165}$$

$$\frac{1}{2}(q'_M)^T HA_n q'_M = u'_{n,M} p'_M \tag{166}$$

$$(q'_M)^T HA_n \Pi_M q'_b = \bar{\rho} c(u'_{n,\Pi} - u'_b) u'_{n,M} + u'_{n,M} p'_\Pi + u'_b p'_M \tag{167}$$

$$-(q'_M)^T HA_n q'_b = -\bar{\rho} c(u'_{n,M} - u'_b) u'_{n,M} - u'_{n,M} p'_M - u'_b p'_M \tag{168}$$

Summing (162)-(168) and substituting this value into the integrand of (159) gives

$$\int_{\partial\Omega_P} E^T HA_n\left(\frac{1}{2}E - E_b\right) dS = \int_{\partial\Omega_P} \bar{\rho} c\left[-(u'_{n,\Pi})^2 - (u'_{n,M})^2 + 2u'_{n,M} u'_{n,\Pi}\right] dS \tag{169}$$

By Young's inequality[28] with $\varepsilon = 1$,

$$2u'_{n,M}u'_{n,\Pi} \leq (u'_{n,M})^2 + (u'_{n,\Pi})^2 \tag{170}$$

Substituting this bound into (169), we have that

$$\int_{\partial\Omega_P} E^T H A_n \left(\frac{1}{2}E - E_b\right) dS \leq \int_{\partial\Omega_P} \bar{\rho} c \left[-(u'_{n,\Pi})^2 - (u'_{n,M})^2 + (u'_{n,M})^2 + (u'_{n,\Pi})^2\right] dS = 0 \tag{171}$$

(171) implies that the first term in the last line of (158) can be omitted, that is,

$$\frac{d}{dt}||E||^2_{(H,\Omega)} \leq -2(W,E)_{(H,\Omega)} \tag{172}$$

Continuing the analysis, note that, for any inner product,

$$(u+v,u+v) = (u,u) + 2(u,v) + (v,v) = ||u||^2 + 2(u,v) + ||v||^2 \geq 0 \tag{173}$$

or

$$-2(u,v) \leq ||u||^2 + ||v||^2 \tag{174}$$

Applying this fact to (172), we have that

$$\frac{d}{dt}||E||^2_{(H,\Omega)} \leq ||E||^2_{(H,\Omega)} + ||W||^2_{(H,\Omega)} \tag{175}$$

By Gronwall's Lemma[29],

$$||E(\cdot,T)||^2_{(H,\Omega)} \quad \leq e^T ||E(\cdot,0)||^2_{(H,\Omega)} + \int_0^T ||W(\cdot,t)||^2_{(H,\Omega)} dt \tag{176}$$

From (157),

$$\begin{aligned} ||W||_{(H,\Omega)} &= \left|\left|\Pi_M\left(A_i\tfrac{\partial q'}{\partial x_i}\right) - A_i\tfrac{\partial(\Pi_M q')}{\partial x_i} - A_i\tfrac{\partial q'}{\partial x_i} + A_i\tfrac{\partial q'}{\partial x_i}\right|\right|_{(H,\Omega)} \\ &\leq \left|\left|\Pi_M\left(A_i\tfrac{\partial q'}{\partial x_i}\right) - A_i\tfrac{\partial q'}{\partial x_i}\right|\right|_{(H,\Omega)} + \left|\left|A_i\tfrac{\partial(\Pi_M q')}{\partial x_i} - A_i\tfrac{\partial q'}{\partial x_i}\right|\right|_{(H,\Omega)} \\ &= ||\Pi_M(\mathscr{L}q') - \mathscr{L}q'||_{(H,\Omega)} + \left|\left|A_i\tfrac{\partial}{\partial x_i}\left(\Pi_M q' - q'\right)\right|\right|_{(H,\Omega)} \\ &= \left|\left|\Pi_M\left(\tfrac{\partial q'}{\partial t}\right) - \tfrac{\partial q'}{\partial t}\right|\right|_{(H,\Omega)} + ||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)} \\ &= \tfrac{\partial}{\partial t}||\Pi_M q' - q'||_{(H,\Omega)} + ||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)} \end{aligned} \tag{177}$$

Before proceeding, let us say a few things about the second term in the last line of (177) that involves the norm of the differential operator $\mathscr{L}$ defined in (151).

6.2.1 Norms Involving the Differential Operator $\mathscr{L}(\cdot)$

Let $\mathscr{W}^1(\Omega)$ be the Sobolev space that results when the vector space $\mathscr{V}$ is equipped with the norm $||\cdot||_{(1;H,\Omega)}$ defined by: for a C^1 function $v \in \mathscr{W}^1(\Omega)$,

$$||v||^2_{(1;H,\Omega)} \equiv ||v||^2_{(H,\Omega)} + \sum_{i=1}^{3}\left|\left|\frac{\partial v}{\partial x_i}\right|\right|^2_{(H,\Omega)} \tag{178}$$

[28] See Section 9.4 of the Appendix.
[29] See Section 9.5 of the Appendix.

We will refer to this norm as the "Sobolev (H,Ω)-norm" (to distinguish it from $||\cdot||_{(H,\Omega)}$, the "Hilbert (H,Ω)-norm"). Now

$$\begin{aligned} ||\mathscr{L}(\Pi_M q' - q')||^2_{(H,\Omega)} &\le \max_{j\in\{1,2,3\}} ||A_j||^2_{(H,\Omega)} \Sigma_{i=1}^3 \left|\left|\frac{\partial}{\partial x_i}(\Pi_M q' - q')\right|\right|^2_{(H,\Omega)} \\ &= K\left(||\Pi_M q' - q'||^2_{(1;H,\Omega)} - ||\Pi_M q' - q'||^2_{(H,\Omega)}\right) \\ &\le K||\Pi_M q' - q'||^2_{(1;H,\Omega)} \end{aligned} \tag{179}$$

where $K = \max_{j\in\{1,2,3\}} ||A_j||^2_{(H,\Omega)}$[30].

One can also bound $||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)}$ using the sub-multiplicativity property of the (H,Ω)-norm:

$$||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)} \le ||\mathscr{L}||_{(H,\Omega)} |\Pi_M q' - q'||_{(H,\Omega)} \tag{180}$$

The norm of $\mathscr{L}$ in the (H,Ω)-norm can be related to the L^2 norm of $D_{x_i} q' \equiv \frac{\partial q'}{\partial x_i}$: for $q' \in \mathscr{H}(\Omega)$,

$$\begin{aligned} ||\mathscr{L} q'||^2_{(H,\Omega)} &= \left|\left|A_i \frac{\partial q'}{\partial x_i}\right|\right|^2_{(H,\Omega)} \\ &= \left|\left|H^{1/2} A_i \frac{\partial q'}{\partial x_i}\right|\right|^2_{L^2(\Omega)} \\ &\le ||H^{1/2}||^2_{L^2(\Omega)} ||A_i||^2_{L^2(\Omega)} \left|\left|\frac{\partial q'}{\partial x_i}\right|\right|^2_{L^2(\Omega)} \\ &\le ||H||_{L^2(\Omega)} ||A_i||^2_{L^2(\Omega)} ||D_{x_i} q'||^2_{L^2(\Omega)} \end{aligned} \tag{181}$$

so that an estimate of $||\mathscr{L}||_{(H,\Omega)}$ is

$$||\mathscr{L}||_{(H,\Omega)} \le ||H||^{1/2}_{L^2(\Omega)} ||A_i||_{L^2(\Omega)} ||D_{x_i}||_{L^2(\Omega)} \tag{182}$$

(note the implied summation on the i's in (182)). It follows that, if one can obtain an estimate of the L^2 norm of the differential operator $D_{x_i} \equiv \frac{\partial}{\partial x_i}$, one can use (182) to estimate the (H,Ω)-norm of $\mathscr{L}$ in (180)[31].

Given the discussion on how to define norms of expressions involving $\mathscr{L}$, (177) can be bounded in two ways, depending on whether or not one wishes to use the Sobolev norm $||\cdot||_{(1;H,\Omega)}$:

$$||W||_{(H,\Omega)} \le \begin{cases} \frac{\partial}{\partial t} ||\Pi_M q' - q'||_{(H,\Omega)} + ||\mathscr{L}||_{(H,\Omega)} ||(\Pi_M q' - q')||_{(H,\Omega)} & \text{(if using Hilbert } (H,\Omega)\text{-norm)} \\ \frac{\partial}{\partial t} ||\Pi_M q' - q'||_{(H,\Omega)} + K^{1/2} ||\Pi_M q' - q'||_{(1;H,\Omega)} & \text{(if using Sobolev } (H,\Omega)\text{-norm)} \end{cases} \tag{183}$$

Here, $K = \max_{i\in\{1,2,3\}} ||A_i||^2_{(H,\Omega)}$.

We are now ready to state and prove the following lemma, which gives a bound on $||(q' - q'_M)(\cdot,T)||_{(H,\Omega)}$ in $\mathscr{H}(\Omega)$.

Lemma 6.2.1. *Let $q' \in \mathscr{H}(\Omega)$ satisfy* (151) *and $q'_M \in \mathscr{H}^M(\Omega)$ satisfy* (154)*. Let $\Pi_M : \mathbb{R}^5 \to \mathscr{V}^M$ be an orthogonal projection operator satisfying properties 1-6 of Section 6.1.1, and let $E \equiv \Pi_M q' - q'_M$. Then*

$$||(q' - q'_M)(\cdot,T)||_{(H,\Omega)} \le e^{\frac{1}{2}T} ||E(\cdot,0)||_{(H,\Omega)} + 2||(\Pi_M q' - q')(\cdot,T)||_{(H,\Omega)} + \int_0^T ||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)} \, dt \tag{184}$$

where

$$||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)} \le \begin{cases} ||\mathscr{L}||_{(H,\Omega)} ||\Pi_M q' - q'||_{(H,\Omega)} & \textit{(if using Hilbert } (H,\Omega)\textit{-norm)} \\ K^{1/2} ||\Pi_M q' - q'||_{(1;H,\Omega)} & \textit{(if using Sobolev } (H,\Omega)\textit{-norm)} \end{cases} \tag{185}$$

[30] See Section 9.6 of the Appendix for a definition of the operator norms of the A_i matrices.

[31] For inequalities involving Sobolev and L^2 norms, refer to Chapter 6 of [9].

Here, $K = \max_{i\in\{1,2,3\}} ||A_i||^2_{(H,\Omega)}$*. The Sobolev* (H,Ω)*-norm is defined in* (178) *and* $||\mathscr{L}||_{(H,\Omega)}$ *is bounded as in e.g.,* (182)*.*

Proof. Note that $q' - q'_M = q' - \Pi_M q' + \Pi_M q' - q'_M = (q' - \Pi_M q') + E$. By the triangle inequality,

$$||(q' - q'_M)(\cdot,T)||_{(H,\Omega)} \le ||(\Pi_M q' - q')(\cdot,T)||_{(H,\Omega)} + ||E(\cdot,T)||_{(H,\Omega)} \tag{186}$$

where $||E(\cdot,T)_{(H,\Omega)}||^2$ is bounded according to (176). From (176) and using the fact that $\int_\Omega f^2 d\Omega \le (\int_\Omega |f| d\Omega)^2$ for some integrand $f : \Omega \to \mathbb{R}$,

$$\begin{aligned} ||E(\cdot,T)||_{(H,\Omega)} &\le \left(e^T ||E(\cdot,0)||^2_{(H,\Omega)} + \int_0^T ||W(\cdot,t)||^2_{(H,\Omega)} dt\right)^{1/2} \\ &\le e^{\frac{1}{2}T} ||E(\cdot,0)||_{(H,\Omega)} + \left(\int_0^T ||W(\cdot,t)||^2_{(H,\Omega)} dt\right)^{1/2} \\ &\le e^{\frac{1}{2}T} ||E(\cdot,0)||_{(H,\Omega)} + \int_0^T ||W(\cdot,t)||_{(H,\Omega)} dt \end{aligned} \tag{187}$$

Now, substituting (177) into (187),

$$\begin{aligned} ||E(\cdot,T)||_{(H,\Omega)} &\le e^{\frac{1}{2}T} ||E(\cdot,0)||_{(H,\Omega)} + \int_0^T \left(\frac{\partial}{\partial t} ||\Pi_M q' - q'||_{(H,\Omega)} + ||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)}\right) dt \\ &\le e^{\frac{1}{2}T} ||E(\cdot,0)||_{(H,\Omega)} + ||(\Pi_M q' - q')(\cdot,T)||_{(H,\Omega)} + \int_0^T ||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)} dt \end{aligned} \tag{188}$$

Substituting (188) into (186) and bounding the term involving $\mathscr{L}$ in the chosen norm following the discussion of Section 6.2.1 gives the desired result. □

Although Lemma 6.2.1 gives a bound for the quantity of interest, namely the error in the ROM solution $||(q' - q'_M)(\cdot,T)||_{(H,\Omega)}$, the estimate (184) is not practically useful, as it contains expressions for which one does not possess any bounds, e.g., $||\Pi_M q' - q'||_{(H,\Omega)}$. It would be useful to relate this expression to quantities that *can* be estimated, at least in theory, such as $||\Pi_M q'_h - q'_h||_{(H,\Omega)}$ (the error between the CFD solution and the projection of the CFD solution onto $\mathscr{V}^M$) and $||q' - q'_h||_{(H,\Omega)}$ (the error in the CFD solution relative to the exact solution).

Thanks to the triangle inequality, it is straight-forward to extend Lemma 6.2.1 into the following theorem, in which the right-hand-side of the error estimate contains only expressions like $||\Pi_M q'_h - q'_h||_{(H,\Omega)}$, $||q' - q'_h||_{(H,\Omega)}$, and $||E(\cdot,0)||_{(H,\Omega)}$, which one should be able to estimate in some way. In this sense, the bound (189) is a "closed" expression for $||(q' - q'_M)(\cdot,T)||_{(H,\Omega)}$.

Theorem 6.2.2. *Let* $q' \in \mathscr{H}(\Omega)$ *satisfy* (151) *and* $q'_M \in \mathscr{H}^M(\Omega)$ *satisfy* (154)*. Let* $\Pi_M : \mathbb{R}^5 \to \mathscr{V}^M$ *be an orthogonal projection operator satisfying properties 1-6 in Section 6.1.1, and let* $E \equiv \Pi_M q' - q'_M$*. Let* $q'_h \in \mathscr{H}^h(\Omega)$ *be the CFD solution. Then*

$$\begin{aligned} ||(q' - q'_M)(\cdot,T)||_{(H,\Omega)} &\le e^{\frac{1}{2}T} ||E(\cdot,0)||_{(H,\Omega)} + 2\left|\left|(q'_h - \Pi_M q'_h)(\cdot,T)\right|\right|_{(H,\Omega)} \\ &\quad +4\left|\left|(q' - q'_h)(\cdot,T)\right|\right|_{(H,\Omega)} + \int_0^T ||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)} dt \end{aligned} \tag{189}$$

where

$$||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)} \le \begin{cases} ||\mathscr{L}||_{(H,\Omega)} \left[||q'_h - \Pi_M q'_h||_{(H,\Omega)} + 2||q' - q'_h||_{(H,\Omega)}\right] & \text{(if using Hilbert } (H,\Omega)\text{-norm)} \\ K^{1/2} \left[||q'_h - \Pi_M q'_h||_{(H,\Omega)} + 2||q' - q'_h||_{(H,\Omega)}\right] & \text{(if using Sobolev } (H,\Omega)\text{-norm)} \end{cases} \tag{190}$$

Here, $K = \max_{i\in\{1,2,3\}} ||A_i||^2_{(H,\Omega)}$*. The Sobolev* (H,Ω)*-norm is defined in* (178) *and* $||\mathscr{L}||_{(H,\Omega)}$ *is bounded as in e.g.,* (182)*.*

Proof. Let $q'_h \in \mathscr{H}^h(\Omega)$ be the CFD solution. By the triangle inequality,

$$\begin{aligned} ||q' - \Pi_M q'||_{(H,\Omega)} &= ||q' - \Pi_M q' + q'_h - \Pi_M q'_h - q'_h + \Pi_M q'_h||_{(H,\Omega)} \\ &\le ||q'_h - \Pi_M q'_h||_{(H,\Omega)} + ||q' - q'_h||_{(H,\Omega)} + ||\Pi_M (q' - q'_h)||_{(H,\Omega)} \\ &\le ||q'_h - \Pi_M q'_h||_{(H,\Omega)} + (1 + ||\Pi_M||_{(H,\Omega)}) ||q' - q'_h||_{(H,\Omega)} \\ &\le ||q'_h - \Pi_M q'_h||_{(H,\Omega)} + 2||q' - q'_h||_{(H,\Omega)} \end{aligned} \tag{191}$$

(using the fact that $||\Pi_M||_{(H,\Omega)} = 1$, Π_M being an orthogonal projector; see Section 6.1.1 above). Applying the triangle inequality to (185),

$$||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)} \leq \begin{cases} ||\mathscr{L}||_{(H,\Omega)} \left[||q'_h - \Pi_M q'_h||_{(H,\Omega)} + 2||q' - q'_h||_{(H,\Omega)}\right] & \text{(if using Hilbert } (H,\Omega)\text{-norm)} \\ K^{1/2}\left[||q'_h - \Pi_M q'_h||_{(H,\Omega)} + 2||q' - q'_h||_{(H,\Omega)}\right] & \text{(if using Sobolev } (H,\Omega)\text{-norm)} \end{cases} \tag{192}$$

Substituting (191) and (192) into (184) and rearranging gives (189). ☐

Remark 14: At first glance, it may appear as though the $\int_0^T ||q'_h - \Pi_M q'_h||_{(H,\Omega)} d\Omega$ term in (189) is simply $||q'_h - \Pi_M q'_h||^{avg}_{(H,\Omega)}$, which is given by (148). However, this is *not* the case, as $||q'_h - \Pi_M q'_h||^{avg}_{(H,\Omega)} \equiv \sqrt{\int_0^T ||q'_h - \Pi_M q'_h||^{\mathbf{2}}_{(H,\Omega)} d\Omega}$, from (140) (in particular, note the exponent in the integrand).

6.3 Error Estimates in the Hilbert Space $\mathscr{H}_{avg}(\Omega)$

Recall from Section 6.1.1 that we have at our disposal an expression for $||q'_h - \Pi_M q'_h||^{avg}_{(H,\Omega)}$, the norm of the difference between q'_h and $\Pi_M q'_h$ in the Hilbert space $\mathscr{H}_{avg}(\Omega)$ in terms of the eigenvalues of the operator $\mathscr{R}$ (146). What more, this expression can be evaluated, as the eigenvalues of $\mathscr{R}$ are computed in determining the POD basis. We cannot use this result in bounding the error in the space $\mathscr{H}(\Omega)$ (Remark 14); however, we *can* use it if we instead bound the error in $\mathscr{H}_{avg}(\Omega)$.

Let us now derive the analogs of Lemma 6.2.1 and Theorem 6.2.2 in the space $\mathscr{H}_{avg}(\Omega)$. Since our goal is to use the estimate (148) which involves a time-average of a Hilbert (H,Ω)-norm (and not the Sobolev (H,Ω)-norm defined in Section 6.2.1), we will use the bound $||\mathscr{L}(\Pi_M q' - q')||_{(H,\Omega)} \leq ||\mathscr{L}||_{(H,\Omega)} ||\Pi_M q' - q'||_{(H,\Omega)}$ from this point forward.

Lemma 6.3.1. *Let* $q' \in \mathscr{H}_{avg}(\Omega)$ *satisfy* (151) *and* $q'_M \in \mathscr{H}^M_{avg}(\Omega)$ *satisfy* (154)*. Let* $\Pi_M : \mathbb{R}^5 \to \mathscr{V}^M$ *be an orthogonal projection operator satisfying properties 1-6 in Section 6.1.1, and let* $E \equiv \Pi_M q' - q'_M$*. Then*

$$||(q' - q'_M)(\cdot,T)||^{avg}_{(H,\Omega)} \leq \frac{1}{\sqrt{T}}(e^T - 1)^{1/2}||E(\cdot,0)||_{(H,\Omega)} + \left[1 + \left(1 + ||\mathscr{L}||_{(H,\Omega)} + T||\mathscr{L}||^2_{(H,\Omega)}\right)^{1/2}\right]||(\Pi_M q' - q')(\cdot,T)||^{avg}_{(H,\Omega)} \tag{193}$$

Proof. As before, since $q' - q'_M = q' - \Pi_M q' + \Pi_M q' - q'_M = (q' - \Pi_M q') + E$, by the triangle inequality,

$$||(q' - q'_M)(\cdot,T)||^{avg}_{(H,\Omega)} \leq ||(\Pi_M q' - q')(\cdot,T)||^{avg}_{(H,\Omega)} + ||E(\cdot,T)||^{avg}_{(H,\Omega)} \tag{194}$$

where $||E(\cdot,T)_{(H,\Omega)}||^2$ is bounded according to (176). From (176),

$$\begin{aligned} ||E(\cdot,T)||^{avg}_{(H,\Omega)} &= \left(\frac{1}{T}\int_0^T ||E(\cdot,\tau)||^2_{(H,\Omega)} d\tau\right)^{1/2} \\ &\leq \left\{\frac{1}{T}\int_0^T \left(e^\tau ||E(\cdot,0)||^2_{(H,\Omega)} + \int_0^\tau ||W(\cdot,t)||^2_{(H,\Omega)} dt\right) d\tau\right\}^{1/2} \\ &= \left\{\frac{1}{T}(e^T - 1)||E(\cdot,0)||^2_{(H,\Omega)} + \frac{1}{T}\int_0^T \int_0^\tau ||W(\cdot,t)||^2_{(H,\Omega)} dt d\tau\right\}^{1/2} \end{aligned} \tag{195}$$

From (177),

$$\begin{aligned} \int_0^\tau ||W(\cdot,t)||^2_{(H,\Omega)} dt &\leq \int_0^\tau \left(\frac{\partial}{\partial t}||\Pi_M q' - q'||_{(H,\Omega)} + ||\mathscr{L}||_{(H,\Omega)} ||\Pi_M q' - q'||_{(H,\Omega)}\right)^2 dt \\ &= \int_0^\tau \left\{\left(\frac{\partial}{\partial t}||\Pi_M q' - q'||_{(H,\Omega)}\right)^2 + ||\mathscr{L}||_{(H,\Omega)} \frac{\partial}{\partial t}||\Pi_M q' - q'||^2_{(H,\Omega)} + ||\mathscr{L}||^2_{(H,\Omega)} ||\Pi_M q' - q'||^2\right\} dt \\ &\leq \left(\int_0^\tau \frac{\partial}{\partial t}||\Pi_M q' - q'||_{(H,\Omega)} dt\right)^2 + ||\mathscr{L}||_{(H,\Omega)} ||(\Pi_M q' - q')(\cdot,\tau)||^2_{(H,\Omega)} + ||\mathscr{L}||^2_{(H,\Omega)} \int_0^\tau ||\Pi_M q' - q'||^2 dt \\ &\leq ||(\Pi_M q' - q')(\cdot,\tau)||^2_{(H,\Omega)} + ||\mathscr{L}||_{(H,\Omega)} ||(\Pi_M q' - q')(\cdot,\tau)||^2_{(H,\Omega)} + ||\mathscr{L}||^2_{(H,\Omega)} \int_0^\tau ||\Pi_M q' - q'||^2 dt \end{aligned} \tag{196}$$

so that

$$\begin{aligned}\frac{1}{T}\int_0^T\int_0^\tau ||W(\cdot,t)||^2_{(H,\Omega)}dt\,d\tau &\leq \frac{1}{T}\int_0^T\left\{\left(1+||\mathscr{L}||_{(H,\Omega)}\right)||(\Pi_M q'-q')(\cdot,\tau)||^2_{(H,\Omega)}+||\mathscr{L}||^2_{(H,\Omega)}\int_0^\tau ||(\Pi_M q'-q')(\cdot,t)||^2 dt\right\}d\tau \\ &\leq \left(1+||\mathscr{L}||_{(H,\Omega)}+T||\mathscr{L}||^2_{(H,\Omega)}\right)\frac{1}{T}\int_0^T ||(\Pi_M q'-q')(\cdot,\tau)||^2_{(H,\Omega)}d\tau \\ &= \left(1+||\mathscr{L}||_{(H,\Omega)}+T||\mathscr{L}||^2_{(H,\Omega)}\right)\left(||(\Pi_M q'-q')(\cdot,T)||^{avg}_{(H,\Omega)}\right)^2\end{aligned} \tag{197}$$

Substituting (197) into (195) gives

$$\begin{aligned}||E(\cdot,T)||^{avg}_{(H,\Omega)} &\leq \left\{\frac{1}{T}(e^T-1)||E(\cdot,0)||^2_{(H,\Omega)}+\left(1+||\mathscr{L}||_{(H,\Omega)}+T||\mathscr{L}||^2_{(H,\Omega)}\right)\left(||(\Pi_M q'-q')(\cdot,T)||^{avg}_{(H,\Omega)}\right)^2\right\}^{1/2} \\ &\leq \frac{1}{\sqrt{T}}(e^T-1)^{1/2}||E(\cdot,0)||_{(H,\Omega)}+\left(1+||\mathscr{L}||_{(H,\Omega)}+T||\mathscr{L}||^2_{(H,\Omega)}\right)^{1/2}||(\Pi_M q'-q')(\cdot,T)||^{avg}_{(H,\Omega)}\end{aligned} \tag{198}$$

Substituting (198) into (194) gives the desired result.

□

As before in the space $\mathscr{H}(\Omega)$, the next step is to relate $||q'-q'_M||^{avg}_{(H,\Omega)}$ to $||q'_h-\Pi_M q'_h||^{avg}_{(H,\Omega)}$ and $||q'-q'_h||^{avg}_{(H,\Omega)}$ using the triangle inequality. The former of these is related to the eigenvalues of $\mathscr{R}$ by (148) and therefore computable.

Theorem 6.3.2. *Let $q'\in\mathscr{H}_{avg}(\Omega)$ satisfy* (151) *and $q'_M\in\mathscr{H}^M_{avg}(\Omega)$ satisfy* (154)*. Let $\Pi_M:\mathbb{R}^5\to\mathscr{V}^M$ be an orthogonal projection operator satisfying properties 1-6 in Section 6.1.1, and let $E\equiv\Pi_M q'-q'_M$. Let $q'_h\in\mathscr{H}^h_{avg}(\Omega)$ be the CFD solution. Then*

$$\begin{aligned}||(q'-q'_M)(\cdot,T)||^{avg}_{(H,\Omega)} &\leq \left[1+\left(1+||\mathscr{L}||_{(H,\Omega)}+T||\mathscr{L}||^2_{(H,\Omega)}\right)^{1/2}\right]\sqrt{\textstyle\sum_{j=M+1}^N\lambda_j} \\ &+2\left[1+\left(1+||\mathscr{L}||_{(H,\Omega)}+T||\mathscr{L}||^2_{(H,\Omega)}\right)^{1/2}\right]||(q'-q'_h)(\cdot,T)||^{avg}_{(H,\Omega)} \\ &+\frac{1}{\sqrt{T}}(e^T-1)^{1/2}||E(\cdot,0)||_{(H,\Omega)}\end{aligned} \tag{199}$$

Here, $\lambda_1\leq\cdots\leq\lambda_M\leq\cdots\leq\lambda_N$ are the ordered eigenvalues of the operator $\mathscr{R}$ defined in (146)*.*

Proof. As in (191), by the triangle inequality,

$$||q'-\Pi_M q'||^{avg}_{(H,\Omega)}\leq||q'_h-\Pi_M q'_h||^{avg}_{(H,\Omega)}+2||q'-q'_h||^{avg}_{(H,\Omega)} \tag{200}$$

Substituting (200) into (193) gives

$$\begin{aligned}||(q'-q'_M)(\cdot,T)||_{(H,\Omega)} &\leq \left[1+\left(1+||\mathscr{L}||_{(H,\Omega)}+T||\mathscr{L}||^2_{(H,\Omega)}\right)^{1/2}\right]\left(||q'_h-\Pi_M q'_h||^{avg}_{(H,\Omega)}+2||q'-q'_h||^{avg}_{(H,\Omega)}\right) \\ &+\frac{1}{\sqrt{T}}(e^T-1)^{1/2}||E(\cdot,0)||_{(H,\Omega)}\end{aligned} \tag{201}$$

Rearranging (201) and substituting (148) in for the $||q'_h-\Pi_M q'_h||^{avg}_{(H,\Omega)}$ term gives (199). □

The results shown in Lemmas 6.2.1 and 6.3.1 and Theorems 6.2.2 and 6.3.2 are convergence estimates. These bounds show that as $q'_h\to q'$ and $\Pi_M q'_h\to q'_h$, $q'_M\to q'$, that is the ROM solution converges to the exact analytical solution to (3). We emphasize that the bound (199) (Theorem 6.3.2) is a *computable*, not merely a theoretical error estimate.

Remark 15: By Remark 13 above, Theorem 6.3.2 is valid as N, the number of snapshots, $\to \infty$. It may be worth examining the validity of substituting the expression $||\Pi_M q_h' - q_h'||_{(H,\Omega)}^{avg} \leftarrow \sqrt{\sum_{j=M+1}^{N} \lambda_j}$, which holds for the discrete time-average norm, for a term that is defined using the continuous time-average norm. Making this substitution would add an additional error term to the bounds in Theorem 6.3.2. It may be possible to quantify this error using Taylor expansions.

7 Extension to Non-Uniform Base Flow $\left(\frac{\partial A_i}{\partial x_j} \neq 0, C \neq 0\right)$

As explained in the Introduction, in this document we have assumed several things about the flow, including that the base flow is uniform. This enables one to neglect the C matrix in (3), as well as omit all terms of the form $\frac{\partial A_i}{\partial x_j}$, $i, j \in \{1,2,3\}$ that arise in integrating the linearized Euler equations, as they are identically zero under the uniform base flow assumption.

The next step in extending the stability analysis of the ROM is to consider the more general case of non-uniform base flow. Then $C \neq 0$ in (3) and $\frac{\partial A_i}{\partial x_j} \neq 0$ in all the derivations performed herein. A natural question to ask is whether this change in assumptions alters the well-posedness and stability of the IBVP (3). In Section 7.1 below, we begin this more general analysis by considering for now only the issue of well-posedness. In particular, we show that if an IBVP assuming uniform base flow is well-posed, then the same IBVP but with non-uniform base flow is also well-posed. This suggests that the well-posedness and stability results shown in this document assuming a uniform base flow will still hold if one considers the more general case of non-uniform base flow.

7.1 Well-Posedness

Consider the following IBVP for the linearized Euler equations, call it "IBVP":

$$IBVP: \begin{cases} \frac{\partial q'}{\partial t} + A_i \frac{\partial q'}{\partial x_i} + Cq' = 0, & \mathbf{x} \in \Omega, \quad 0 < t < T \\ Pq' = h, & \mathbf{x} \in \partial\Omega_P, \quad 0 < t < T \\ q'(\mathbf{x}, 0) = f(\mathbf{x}), & \mathbf{x} \in \Omega \end{cases} \tag{202}$$

(note that, as before, we are neglecting the far-field boundary conditions, assuming they are well-posed). Suppose we have a non-uniform base flow, so that $C \neq 0$, $\frac{\partial A_i}{\partial x_j} \neq 0$ and $\frac{\partial H}{\partial x_j} \neq 0$, $i, j \in \{1,2,3\}$. Then the following result holds.

Theorem 7.1.1. *Let IBVP* be the IBVP corresponding to* (202) *but assuming uniform base flow:*

$$IBVP^*: \begin{cases} \frac{\partial q'}{\partial t} + A_i \frac{\partial q'}{\partial x_i} = 0, & \mathbf{x} \in \Omega, \quad 0 < t < T \\ Pq' = h, & \mathbf{x} \in \partial\Omega_P, \quad 0 < t < T \\ q'(\mathbf{x}, 0) = f(\mathbf{x}), & \mathbf{x} \in \Omega \end{cases} \tag{203}$$

where, for IBVP*, $\frac{\partial A_i}{\partial x_j} = \frac{\partial H}{\partial x_j} \equiv 0$, $i, j \in \{1,2,3\}$. Suppose the boundary condition $Pq' = h$ is well-posed for IBVP* with

$$\frac{d}{dt}||q'||_{(H,\Omega)}^2 \leq 0 \tag{204}$$

under the assumption of uniform base flow. Then the problem IBVP in (202) *is well-posed with the energy estimate:*

$$||q'(\cdot, T)||_{(H,\Omega)}^2 \leq e^{h(\bar{q}, \nabla \bar{q})T} ||f(\cdot)||_{(H,\Omega)}^2 \tag{205}$$

where

$$h(\overline{q},\nabla\overline{q}) = \left|\left|\frac{\partial A_i}{\partial x_i}+H^{-1}\frac{\partial H}{\partial x_i}A_i-2C\right|\right|_{(H,\Omega)} = \left\{\int_\Omega\left(\frac{\partial A_i}{\partial x_i}+H^{-1}\frac{\partial H}{\partial x_i}A_i-2C\right)^T H\left(\frac{\partial A_i}{\partial x_i}+H^{-1}\frac{\partial H}{\partial x_i}A_i-2C\right)d\Omega\right\}^{1/2} \tag{206}$$

(Note that $h(\overline{q},\nabla\overline{q})\geq 0$).

Proof. For $IBVP^*$,

$$\frac{1}{2}\frac{d}{dt}||q'||^2_{(H,\Omega)} = -\frac{1}{2}\int_{\partial\Omega} q'^T H A_n q' dS \leq 0 \tag{207}$$

by the hypothesis (204). Now consider $IBVP$. For non-uniform base flow,

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}||q'||^2_{(H,\Omega)} &= \textstyle\int_\Omega q'^T H\frac{\partial q'}{\partial t}d\Omega\\
&= -\textstyle\int_\Omega q'^T H\left[A_i\frac{\partial q'}{\partial x_i}+Cq'\right]d\Omega\\
&= -\tfrac{1}{2}\textstyle\int_\Omega \frac{\partial}{\partial x_i}[q'^T H A_i q']d\Omega + \tfrac{1}{2}\int_\Omega q'^T\frac{\partial(HA_i)}{\partial x_i}q'd\Omega - \int_\Omega q'^T HCq'd\Omega\\
&= \underbrace{-\frac{1}{2}\int_{\partial\Omega_P} q'^T H A_n q' dS}_{\leq 0 \text{ by } (207)} + \textstyle\int_\Omega q'^T H\left[\frac{1}{2}H^{-1}\frac{\partial(HA_i)}{\partial x_i}-C\right]q'd\Omega\\
&= \left(\left[\tfrac{1}{2}H^{-1}\tfrac{\partial(HA_i)}{\partial x_i}-C\right]q',q'\right)_{(H,\Omega)}\\
&\leq \left|\left(\left[\tfrac{1}{2}H^{-1}\tfrac{\partial(HA_i)}{\partial x_i}-C\right]q',q'\right)_{(H,\Omega)}\right|\\
&\leq \tfrac{1}{2}\left|\left|\tfrac{\partial A_i}{\partial x_i}+H^{-1}\tfrac{\partial H}{\partial x_i}A_i-2C\right|\right|_{(H,\Omega)}||q'||^2_{(H,\Omega)}\\
&= \tfrac{1}{2}h(\overline{q},\nabla\overline{q})||q'||^2_{(H,\Omega)}
\end{aligned} \tag{208}$$

In going from line 6 to line 7 in (208), we have applied the Cauchy-Schwarz inequality (see Section 9.2 of the Appendix). Now, by Gronwall's lemma (Section 9.5 of the Appendix),

$$||q'(\cdot,T)||^2_{(H,\Omega)} \leq e^{h(\overline{q},\nabla\overline{q})T}||f(\cdot)||^2_{(H,\Omega)} \tag{209}$$

According to Definition 2.8 in [14] (Section 9.9 of the Appendix), (209) implies $IBVP^*$ is well-posed. □

The estimate (209) shows well-posedness according to Definition 2.8 in [14], with $\alpha = h(\overline{q},\nabla\overline{q})\geq 0$. The proof of Theorem 7.1.1 compared with the proof of Theorem 2.2.1 above suggests that, in the well-posedness energy estimate (233), $\alpha = 0$ when the base flow is uniform, whereas $\alpha\neq 0$ when the base flow is non-uniform.

An extensive study of the stability of the linearized Euler equations (3) under the more general case of non-uniform base flow goes beyond the scope of this work. The preliminary analysis of well-posedness performed in Section 7.1 suggests that the stability results proven herein assuming uniform base flow will carry over to the non-uniform base flow case.

8 Conclusions and Future Work

The analysis presented in this document has shed a great deal of light on well-posedness, stability and convergence issues in the context of Reduced Order Models. The acoustically-reflecting boundary condition (47) and its relation to the penalty method is now well understood. It is particularly reassuring that different analyses of this boundary condition lead to the same stability result and equivalent penalty-like formulations. Convergence estimates and error bounds of the type derived in Section 6 are, to the author's knowledge, novel in the area of Reduced Order Modeling. These bounds combine techniques found in [10], [18], and [22], and the reader is referred to these sources to better understand and/or extend the analysis in Section 6. The text [14] is recommended for a thorough discussion of stability and well-posedness; Chapter 3 of [16] is recommended for an overview of the Proper Orthogonal Decomposition (POD) and reduced order models.

It is worth noting that the analysis presented here has led to several unanswered questions that should be addressed in the future. For one, it is still not entirely clear why weak implementation of the old no-penetration boundary condition (Algorithm 1) did not properly enforce $u'_n = u'_b$ at the plate. We showed in Section 2.5 that this condition is stable for the fluid ROM (neutrally stable, but stable nonetheless), and also that it is mathematically equivalent to the new acoustically-reflecting boundary condition, which is enforced in the same weak fashion. One has yet to come up with a precise mathematical explanation for exactly why this implementation seems to be "too weak" for the no-penetration condition (41).

Another issue that merits further thought, highlighted in Remarks 13 and 15, is the issue of substituting the expression $||\Pi_M q'_h - q'_h||^2 = \sum_{j=M+1}^{N} \lambda_j$, valid for the discrete time-average norm, for a term that is defined using the continuous time-average norm. One would expect that making this substitution would add an additional error component to the error estimates in Section 6. One may try to quantify these using, for instance, Taylor's theorem with remainder.

Besides addressing these and other questions that remain open in light of the preceding analysis, future work should focus on loosening the assumptions on which the derivations presented herein rely. The first step would be to look at the more general case of non-uniform base flow, as we began to do in Section 7. Ultimately, one would like to extend this analysis (and the Reduced Order Model) to the non-linear Euler equations.

9 Appendix

9.1 Integration by Parts "Trick"

Let $G \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $u \in \mathbb{R}^n$ be a vector. Then

$$u^T G \frac{\partial u}{\partial x} = \frac{1}{2}\left[\frac{\partial}{\partial x}(u^T G u) - u^T \frac{\partial G}{\partial x} u\right] \tag{210}$$

9.2 Hölder Inequality

Let Ω be a bounded region in $\mathbb{R}^n$ and suppose $f \in L_p(\Omega)$ and $g \in L_q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_\Omega |fg| d\Omega \leq \left(\int_\Omega |f|^p d\Omega\right)^{1/p} \left(\int_\Omega |g|^q d\Omega\right)^{1/q} \tag{211}$$

or, using norm notation,

$$||fg||_1 \leq ||f||_p ||g||_q \tag{212}$$

The Hölder inequality with $p = q = 2$ is the *Cauchy-Schwarz inequality*.

9.3 Minkowski Inequality

Let Ω be a bounded region in $\mathbb{R}^n$ and suppose $f \in L_p(\Omega)$ and $g \in L_p(\Omega)$ with $1 \leq p \leq \infty$. Then

$$\left(\int_\Omega |f+g|^p d\Omega\right)^{1/p} \leq \left(\int_\Omega |f|^p d\Omega\right)^{1/p} + \left(\int_\Omega |g|^p d\Omega\right)^{1/p} \tag{213}$$

or, using norm notation,

$$||f+g||_p \leq ||f||_p + ||g||_p \tag{214}$$

9.4 Young's Inequality

Let a and b be non-negative real numbers and let $\varepsilon > 0$. Then

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2} \tag{215}$$

9.5 Gronwall's Lemma

Let I denote an interval of the real line of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with $a < b$. Let β and u be real-valued continuous functions defined on I. If u is differentiable in the interior I_0 of I and satisfies the differential inequality

$$u'(t) \leq \beta(t)u(t), \qquad t \in I_0 \tag{216}$$

then

$$u(t) \leq u(a)\exp\left(\int_a^t \beta(s)ds\right) \tag{217}$$

for all $t \in I$. Note that there are no assumptions on the signs of the functions β and u.

9.6 Operator Norms

In general, the definition of an operator norm $||\cdot||_{op}$ on some normed space X is, for a map A on X is

$$||A||_{op} = \min\{c \in \mathbb{R} : ||Ax|| \leq c||x||, \text{ for all } x \in X\} \tag{218}$$

9.7 Symmetrizer of a Matrix

The following lemma is quoted from [13] (Lemma 6.1.1, p. 211). It gives a sufficient condition for there to exist a symmetrizer H for the first order linear system

$$u_t = Au_x \tag{219}$$

where A is an $n \times n$ constant diagonalizable matrix with real eigenvalues.

Lemma 6.1.1 in [13]. *Let A be a real matrix with real eigenvalues and a complete set of eigenvectors that are the columns of a matrix S. Let D be a real positive diagonal matrix. Then*

$$H \equiv (S_1^{-1})^* D S_1^{-1} \tag{220}$$

is positive definite and Hermitian, and HA is Hermitian; that is, H "symmetrizes" A.

9.7.1 Symmetrizability of Linear Systems of PDEs

All hyperbolic systems of conservation laws arising in continuum physics are symmetrizable (see Chapter 6 of [11]). This is not a mere coincidence, but rather a result of enforcing the second law of thermodynamics by judicial selection of the equations. In the field of fluid mechanics, symmetrizable systems include, for example, the shallow water equations and the linearized Euler equation. For a detailed discussion on deriving symmetrizers for linear systems of PDEs, see [1]. Chapter 6 of [13] may also be of interest.

9.7.2 Application to the Linearized Euler Equations in the Original Variables

Lemma 6.1.1 in [13] can be easily applied to derive the symmetrizer H of the matrices A_1, A_2 and A_3 that arise in the linearization of the Euler equations (3). Here we use the theorem to derive the symmetrizer H given in [2].

Recall that

$$S = \begin{pmatrix} 0 & n_3 & n_2 & \frac{1}{2}n_1 & -\frac{1}{2}n_1 \\ n_3 & 0 & -n_1 & \frac{1}{2}n_2 & -\frac{1}{2}n_2 \\ -n_2 & -n_1 & 0 & \frac{1}{2}n_3 & -\frac{1}{2}n_3 \\ n_1 & -n_2 & n_3 & -\frac{\bar{\zeta}}{2c} & -\frac{\bar{\zeta}}{2c} \\ 0 & 0 & 0 & \frac{\gamma\bar{p}}{2c} & \frac{\gamma\bar{p}}{2c} \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 0 & n_3 & -n_2 & n_1 & \frac{\bar{\zeta}}{\gamma\bar{p}}n_1 \\ n_3 & 0 & -n_1 & -n_2 & -\frac{\bar{\zeta}}{\gamma\bar{p}}n_2 \\ n_2 & -n_1 & 0 & n_3 & \frac{\bar{\zeta}}{\gamma\bar{p}}n_3 \\ n_1 & n_2 & n_3 & 0 & \frac{c}{\gamma\bar{p}} \\ -n_1 & -n_2 & -n_3 & 0 & \frac{c}{\gamma\bar{p}} \end{pmatrix} \tag{221}$$

$$\Lambda = \begin{pmatrix} \bar{u}_n & & & & \\ & \bar{u}_n & & & \\ & & \bar{u}_n & & \\ & & & \bar{u}_n + c & \\ & & & & \bar{u}_n - c \end{pmatrix} \tag{222}$$

where $\bar{u}_n = \bar{u}n_1 + \bar{v}n_2 + \bar{w}n_3$. Take $A_3 = S_3 \Lambda_3 S_3^{-1}$ with $\mathbf{n}^T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ in (221) and (222). To derive the entries of D, write

$$D = \begin{pmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & d_4 & \\ & & & & d_5 \end{pmatrix} \tag{223}$$

Then

$$H = (S_3^{-1})^T D S_3^{-1} = \begin{pmatrix} d_2 & & & & \\ & d_1 & & & \\ & & d_4 + d_5 & & \frac{c}{\gamma\bar{p}}(d_4 - d_5) \\ & & & d_3 & \frac{1}{\gamma\bar{\rho}\bar{p}}d_3 \\ & & \frac{c}{\gamma\bar{p}}(d_4 - d_5) & \frac{1}{\gamma\bar{\rho}\bar{p}}d_3 & \frac{1}{\gamma^2\bar{p}^2}\left[\frac{1}{\bar{\rho}^2}d_3 + c^2(d_4 + d_5)\right] \end{pmatrix} \tag{224}$$

To recover the H given in [2], let $d_1 = d_2 = \bar{\rho}$, $d_3 = \alpha^2\gamma\bar{\rho}^2\bar{p}$, and $d_4 = d_5 = \frac{1}{2}\bar{\rho}$:

$$D = \begin{pmatrix} \bar{\rho} & & & & \\ & \bar{\rho} & & & \\ & & \alpha^2\gamma\bar{\rho}^2\bar{p} & & \\ & & & \frac{1}{2}\bar{\rho} & \\ & & & & \frac{1}{2}\bar{\rho} \end{pmatrix} \tag{225}$$

Then

$$H = \begin{pmatrix} \bar{\rho} & & & & \\ & \bar{\rho} & & & \\ & & \bar{\rho} & & \\ & & & \alpha^2\gamma\bar{\rho}^2\bar{p} & \bar{\rho}\alpha^2 \\ & & & \bar{\rho}\alpha^2 & \frac{1+\alpha^2}{\gamma\bar{p}} \end{pmatrix} \tag{226}$$

One can check that with H given by (226), HA_i for $i = 1,2,3$ are all symmetric. (226) is exactly the symmetrizer H given in [2]. Thus we have derived this symmetrizer with the help of Lemma 6.1.1 in [13]. One could similarly derive symmetrizers by specifying different normal vectors **n** in (221) and (222) [thus, the symmetrizer of the system is not unique].

9.8 Application to the Linearized Euler Equations in the Characteristic Variables

We now derive the symmetrizer of the linearized Euler equations in the characteristic variables $V' = S^{-1}q'$:

$$\frac{\partial V'}{\partial t} + S^{-1}A_iS\frac{\partial V'}{\partial x_i} = 0 \tag{227}$$

Then

$$S^{-1}A_1S = \begin{pmatrix} \bar{u} & 0 & 0 & 0 & 0 \\ 0 & \bar{u} & 0 & \frac{1}{2}cn_3 & \frac{1}{2}cn_3 \\ 0 & 0 & \bar{u} & \frac{1}{2}cn_2 & \frac{1}{2}cn_2 \\ 0 & cn_3 & cn_2 & \bar{u}+cn_1 & 0 \\ 0 & cn_3 & cn_2 & 0 & \bar{u}-cn_1 \end{pmatrix}, \quad S^{-1}A_2S = \begin{pmatrix} \bar{v} & 0 & 0 & \frac{1}{2}cn_3 & \frac{1}{2}cn_3 \\ 0 & \bar{v} & 0 & 0 & 0 \\ 0 & 0 & \bar{v} & -\frac{1}{2}cn_1 & -\frac{1}{2}cn_1 \\ cn_3 & 0 & -cn_1 & \bar{v}+cn_2 & 0 \\ cn_3 & 0 & -cn_1 & 0 & \bar{v}-cn_2 \end{pmatrix},$$
$$S^{-1}A_3S = \begin{pmatrix} \bar{w} & 0 & 0 & -\frac{1}{2}cn_2 & -\frac{1}{2}cn_2 \\ 0 & \bar{w} & 0 & -\frac{1}{2}cn_1 & -\frac{1}{2}cn_1 \\ 0 & 0 & \bar{w} & 0 & 0 \\ -cn_2 & -cn_1 & 0 & \bar{w}+cn_3 & 0 \\ -cn_2 & -cn_1 & 0 & 0 & \bar{w}-cn_3 \end{pmatrix} \tag{228}$$

Although one can apply Lemma 6.1.1 in [13] to find the matrix that symmetrizes $\{A_i^S : i = 1,2,3\}$ simultaneously, it is easier to do this by inspection. Observe that if one lets

$$Q = \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \tag{229}$$

then, denoting $A_i^S \equiv QS^{-1}A_iS$ for $i = 1,2,3$,

$$A_1^S = \begin{pmatrix} 2\overline{u} & 0 & 0 & 0 & 0 \\ 0 & 2\overline{u} & 0 & cn_3 & cn_3 \\ 0 & 0 & 2\overline{u} & cn_2 & cn_2 \\ 0 & cn_3 & cn_2 & \overline{u}+cn_1 & 0 \\ 0 & cn_3 & cn_2 & 0 & \overline{u}-cn_1 \end{pmatrix}, \quad A_2^S = \begin{pmatrix} 2\overline{v} & 0 & 0 & cn_3 & cn_3 \\ 0 & 2\overline{v} & 0 & 0 & 0 \\ 0 & 0 & 2\overline{v} & -cn_1 & -cn_1 \\ cn_3 & 0 & -cn_1 & \overline{v}+cn_2 & 0 \\ cn_3 & 0 & -cn_1 & 0 & \overline{v}-cn_2 \end{pmatrix},$$
$$A_3^S = \begin{pmatrix} 2\overline{w} & 0 & 0 & -cn_2 & -cn_2 \\ 0 & 2\overline{w} & 0 & -cn_1 & -cn_1 \\ 0 & 0 & 2\overline{w} & 0 & 0 \\ -cn_2 & -cn_1 & 0 & \overline{w}+cn_3 & 0 \\ -cn_2 & -cn_1 & 0 & 0 & \overline{w}-cn_3 \end{pmatrix} \tag{230}$$

In particular, each of the A_i^S in (230) are symmetric. It follows that Q symmetrizes $\{S^{-1}A_iS : i = 1,2,3\}$. Not only is the matrix Q symmetric and positive definite, it has the added benefit of being diagonal.

9.9 Well-Posedness

Consider a general initial-boundary value problem (IBVP) of the form

$$\begin{aligned} \tfrac{\partial u}{\partial t} &= Pu + F, \quad t \geq 0 \\ Bu &= g \\ u &= f, \qquad t = 0 \end{aligned} \tag{231}$$

Here, P is a differential operator in space, and B is a boundary operator acting on the solution at the spatial boundary.

The usual "strong" definition of well-posedness for an IBVP (232) is as follows (Definition 2.9 on p. 32 of [14]):

Definition 2.9 in [14]. *The IBVP* (231) *is strongly well-posed if there is a unique solution satisfying*

$$||u(\cdot,t)||^2 \leq Ke^{\alpha t}\left(||f(\cdot)||^2 + \int_0^t ||F(\cdot,\tau)||^2 + |g(\tau)|^2)d\tau\right) \tag{232}$$

where K and α are constants independent of $f(x)$, $F(x,t)$ and $g(t)$.

A weaker definition of well-posedness is Definition 2.8 on p. 32 of [14]:

Definition 2.8 in [14]. *The IBVP* (231) *is well-posed if for $F = 0$, $g = 0$, there is a unique solution satisfying*

$$||u(\cdot,t)|| \leq Ke^{\alpha t}||f(\cdot)|| \tag{233}$$

where K and α are constants independent of $f(x)$.

It is common to use the energy method to check well-posedness. The quantity $\frac{d}{dt}||u||^2$ is an energy measure. Clearly, if

$$\frac{d}{dt}||u(\cdot,t)||^2 \leq 0 \tag{234}$$

then (integrating both sides of (234)) $||u(\cdot,t)||^2 \leq K = const$, meaning (233) is satisfied, so that (231) is well-posed.

9.10 Stability

9.10.1 Definitions

Consider the following semi-discrete problem:

$$\begin{array}{ll} \frac{du_j}{dt} = Qu_j + F_j, & j = 1,2,\ldots,N-1 \\ B_h u = g(t) & \\ u_j(0) = f_j, & j = 1,2,\ldots,N \end{array} \tag{235}$$

where Q is a discretizing operator, F_j and f_j are the discretized version of F and f respectively, and $B_h u$ denotes the complete set of discretized boundary conditions.

Let $||\cdot||_h$ be a discrete norm. The following is the strongest definition of stability (Definition 2.12 on p. 37 of [14]):

Definition 2.12 in [14]. *The semi-discrete IBVP* (235) *is strongly stable if there is a unique solution satisfying*

$$||u(\cdot,t)||_h^2 \leq Ke^{\alpha t}\left(||f(\cdot)||_h^2 + \int_0^t ||F(\cdot,\tau)||_h^2 + |g(\tau)|^2)d\tau\right) \tag{236}$$

where K and α are constants independent of f, F and g.

A weaker definition of stability is Definition 2.11 on p. 37 of [14]:

Definition 2.11 in [14]. *The semi-discrete IBVP* (235) *is stable if there is a unique solution satisfying*

$$||u(\cdot,t)||_h \leq Ke^{\alpha t}||f(\cdot)||_h \tag{237}$$

where K and α are constants independent of f and g.

As with well-posedness, it is common to use energy estimates to check for stability: if

$$\frac{d}{dt}||u(\cdot,t)||_h^2 \leq 0 \tag{238}$$

then (237) is satisfied and we have stability.

9.10.2 Energy Matrix Analysis of Coupled Fluid/Structure Systems ([20])

The following results, presented in [20], are useful in studying the stability of coupled fluid/structure systems such as (121). These results were used to prove stability of the coupled system under the old no-penetration boundary condition at the plate in [17].

Definition 3.1 in [20]. *We say that K is 'stable' if and only if:*

1. *K is diagonalizable in $\mathbb{C}$.*
2. *$\forall\lambda \in Sp(K), \mathscr{R}(\lambda) \leq 0$.*

Theorem 3.1 in [20]. *A real, symmetric positive definite (RSPD) matrix E_K is an energy matrix for K if and only if for all X that solve $\dot{X} = KX$, $\frac{1}{2}\frac{d}{dt}\left(X^T E_K X\right) \leq 0$.*

Theorem 3.4 in [20]. *If A and D are two real, stable matrices with energy matrices E_A and E_D, then*

$$\left\{E_A B + (E_D C)^T = 0\right\} \Rightarrow \left\{K = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ is a stable matrix.}\right\} \tag{239}$$

9.10.3 Lyapunov Stability Condition

A continuous-time linear time-invariant system $\dot{X} = AX$ is Lyapunov stable if and only if all the eigenvalues of A have real parts less than or equal to 0, and those with real parts equal to 0 are non-repeated.

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References

[1] S. Adhikari, On Symmetrizable Systems of Second Kind, *J. Appl. Mech.*, **67**(4) (2000) 797–802.

[2] M.F. Barone, D.J. Segalman, H.K. Thornquist, Galerkin Reduced Order Models for Compressible Flow, *Abstract for the 46th AIAA Aerospace Sciences Meeting and Exhibit Preferred Session: Grouped with Other Papers on Reduced Order Models*, Sandia National Laboratories, Albuquerque, NM (2007).

[3] M.F. Barone, J.L. Payne, Methods for Simulation-based Analysis of Fluid-Structure Interaction, *SAND2005-6573*, Sandia National Laboratories, Albuquerque, NM (2005).

[4] M.F. Barone, I. Kalashnikova, D.J. Segalman, H.K. Thornquist, Galerkin Reduced Order Models for Compressible Flow with Structural Interaction, *AIAA 46th Aerospace Science Meeting and Exhibit*, AIAA 2008-0612, Reno, NV (2008).

[5] D.J. Bodony, Analysis of Sponge Zones for Computational Fluid Mechanics, *J. Comput. Phys.*, **212** (2006) 681–702.

[6] T. Bui-Thanh, K. Willcox, O. Ghattas, Model Reduction for Large-Scale Systems with High-Dimensional Parametric Input Space, *ACDL Technical Report*, TR-07-2 (2007).

[7] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods: Fundamentals in Single Domains, Leipzig, Germany: Springer-Verlag, 2006.

[8] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods: Evolution of Complex Geometries and Applications to Fluid Dynamics, Leipzig, Germany: Springer-Verlag, 2007.

[9] L.C. Evans, Partial Differential Equations, Providence, RI: American Mathematical Society, 1998.

[10] D. Funaro, D. Gottlieb, Convergence results for Pseudospectral Approximations of Hyperbolic Systems by a Penalty-Type Boundary Treatment, *Math. of Comput.* **57**(196) (1991) 585–596.

[11] A. Greven, G. Keller, G. Warnecke, Entropy, Princeton, NJ: Princeton University Press, 2003.

[12] B. Gustafsson, A. Sundstrom, Incompletely Parabolic Problems in Fluid Dynamics, *SIAM J. Appl. Math.*, **35**(2) (1978) 343–357.

[13] B. Gustafsson, H.O. Kreiss, J. Oliger, Time Dependent Problems and Difference Methods, New York, NY: John Wiley & Sons, Inc., 1995.

[14] B. Gustafsson, High Order Difference Methods for Time Dependent PDE, Leipzig, Germany: Springer-Verlag, 2008.

[15] J.S. Hesthaven, D. Gottlieb, A Stable Method for the Compressible Navier-Stokes Equations: I. Open Boundary Conditions, *SIAM J. Sci. Comput.*, **17**(3) (1996) 579–612.

[16] P. Holmes, J.L. Lumley, G. Berkooz, Turbulence, Coherent Structures, Dynamical Systems and Symmetry, New York, NY: Cambridge University Press, 1996.

[17] I. Kalashnikova, Galerkin Reduced Order Model (ROM) for Compressible Flow: Enforcement of Boundary Conditions and Analysis of Coupled Fluid/Structure System, *SAND 5255161*, Sandia National Laboratories, Albuquerque, NM (2007).

[18] K. Kunisch, S. Volkwein, Galerkin Proper Orthogonal Decomposition for a General Equation in Fluid Dynamics, *SIAM J. Numer. Anal.*, **40**(2)(2002) 492–515.

[19] J. Oliger, A. Sundstrom, Theoretical and Practical Aspects of some Initial Boundary Value Problems in Fluid Dynamics, *SIAM J. Appl. Math.* **35**(3)(1978) 419–446.

[20] S. Piperno, C. Farhat, B. Larroututou, Partitioned procedured for the transient solution of coupled aeroelastic problems, *Comp. Methods Appl. Mech. Engrg.* **124** (1995) 79–112.

[21] S. Piperno, C. Farhat, Partitioned procedures for the transient solution of coupled aeroelastic problems - Part II: energy transfer analysis and three-dimensional applications, *Comp. Methods Appl. Mech. Engrg.* **190** (2001) 3147–3170.

[22] M. Rathinam, L.R. Petzold, L.R. A New Look at Proper Orthogonal Decomposition, *SIAM J. Numer. Anal.* **41** (5) (2003) 1893–1925.

[23] J.C. Strikwerda, Finite Difference Schemes and Partial Differential Equations, Pacific Grove, CA: Society for Industrial and Applied Mathematics, 2004.